9.3 Differential topology

9.3.1 Introduction

Differential topology is the study of smooth manifolds and smooth functions among them. In this short tutorial, we avoid needing to define those terms, by focusing on a special case: open subsets of $\mathbb{R}^3$.

Like algebraic topology, differential topology involves a bunch of big machinery, which we don't have time to build. One of the big machines is called de Rham cohomology. For our special case of open subsets of $\mathbb{R}^3$, we can glimpse the meaning and usefulness of de Rham cohomology by doing ordinary multivariable calculus.

9.3.2 Framework

Let X be an open subset of $\mathbb{R}^3$. Thus X is a 3-manifold. In fact, it's a smooth manifold, which basically means "a safe space to do calculus", but don't worry about that. Let

$$\mathcal{S} = \{f : X \to \mathbb{R} \mid f \text{ is smooth}\}$$

be the set of smooth scalar fields on X. Here, "smooth" means C^∞; it means that the function has derivatives of all orders. Let

$$\mathcal{V} = \{\vec{F} : X \to \mathbb{R}^3 \mid \vec{F} \text{ is smooth}\}$$

be the set of all smooth vector fields on X. To say that a vector field $\vec{F} = \langle F_1, F_2, F_3 \rangle$ is smooth is to say that each scalar field $F_i : X \to \mathbb{R}$, that makes up $\vec{F}$, is smooth. Notice that $\mathcal{S}$ and $\mathcal{V}$ are vector spaces. Their dimensions are infinite.

We can now define the gradient function $\operatorname{grad} : \mathcal{S} \to \mathcal{V}$ by

$$\operatorname{grad} f = \left\langle \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3} \right\rangle.$$

And we can define the divergence function $\operatorname{div} : \mathcal{V} \to \mathcal{S}$ by

$$\operatorname{div} \vec{F} = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \frac{\partial F_3}{\partial x_3}.$$

Gradient and divergence can be defined in any dimension, but what comes next is specific to three dimensions. Define the curl function $\operatorname{curl} : \mathcal{V} \to \mathcal{V}$ by

$$\operatorname{curl} \vec{F} = \left\langle \frac{\partial F_3}{\partial x_2} - \frac{\partial F_2}{\partial x_3}, \frac{\partial F_1}{\partial x_3} - \frac{\partial F_3}{\partial x_1}, \frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2} \right\rangle.$$

(These functions are ripe with geometric meaning. At each $\vec{x} \in X$, the gradient tells you how to "climb the hill" defined by f as quickly as possible, the divergence tells you whether $\vec{F}$ "spreads out" or "comes together", and the curl tells you how much $\vec{F}$ "rotates". These functions also have many applications, such as Maxwell's equations of electromagnetism and the Navier-Stokes equations of fluid dynamics.)

While we're at it, let's define a function $\operatorname{const} : \mathbb{R} \to \mathcal{S}$ as follows. For any $c \in \mathbb{R}$, $\operatorname{const}(c)$ is the constant function $f : X \to \mathbb{R}$ defined by $f(\vec{x}) = c$. Then we have the following diagram of vector spaces and functions among them:

$$\mathbb{R} \xrightarrow{\text{const}} \mathcal{S} \xrightarrow{\text{grad}} \mathcal{V} \xrightarrow{\text{curl}} \mathcal{V} \xrightarrow{\text{div}} \mathcal{S}.$$

In fact, all four functions are linear transformations, as you can check. The following exercise lists some important basic relationships among their kernels and images.

Exercise 9.3.1. *For this exercise, it's helpful to recall Clairaut's theorem: If f is C^∞ (and thus C^2), then $\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} f = \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_i} f$ for all $i, j \in \{1, 2, 3\}$.*

1. *Prove that* $\operatorname{im}\operatorname{const} \subseteq \ker\operatorname{grad}$.
2. *Prove that* $\operatorname{im}\operatorname{grad} \subseteq \ker\operatorname{curl}$.
3. *Prove that* $\operatorname{im}\operatorname{curl} \subseteq \ker\operatorname{div}$.
4. *What are* $\ker\operatorname{const}$ *and* $\operatorname{im}\operatorname{div}$*?*

In the preceding exercise, are the subsets proper? Or are some of the containments actually equalities? Surprisingly, the answer turns out to depend on the topology of X — and nothing but the topology. The next few sections explore how.

9.3.3 grad **versus** const

Exercise 9.3.2. *Which topological property of X controls whether* $\operatorname{im}\operatorname{const} = \ker\operatorname{grad}$*? When that property does not hold, how does its failure let us create functions* $f \in \ker\operatorname{grad} - \operatorname{im}\operatorname{const}$*?*

It is useful to rephrase our discoveries in terms of quotients of vector spaces. Define an equivalence relation on $\ker\operatorname{grad}$ by declaring that $f \sim g$ if $f - g \in \operatorname{im}\operatorname{const}$. The set of equivalence classes forms a vector space, which is denoted $\ker\operatorname{grad} / \operatorname{im}\operatorname{const}$. You can check that addition $[f] + [g] = [f + g]$ and scaling $c[f] = [cf]$ are well defined.

Exercise 9.3.3. *What is the dimension of* $\ker\operatorname{grad} / \operatorname{im}\operatorname{const}$*, in terms of topological properties of X?*

9.3.4 curl **versus** grad

The discrepancy between $\operatorname{im}\operatorname{grad}$ and $\ker\operatorname{curl}$ (if any) is harder to discern than the discrepancy between $\operatorname{im}\operatorname{const}$ and $\ker\operatorname{grad}$. So I'm going to guide you through the answer. It turns out that the crucial example is

$$\vec{F} = \left\langle \frac{-x_2}{x_1^2 + x_2^2}, \frac{x_1}{x_1^2 + x_2^2}, 0 \right\rangle$$

(or any scalar multiple of that $\vec{F}$).

Exercise 9.3.4. *On what domain X is that $\vec{F}$ defined? Check that* $\vec{F} \in \ker\operatorname{curl}$ *on that domain.*

In calculus, the anti-derivative $\int g(u)\, du$ of a function g is unique up to an additive constant (the infamous "$+\, C$"). Similarly, if there exists a scalar field f such that $\vec{F} = \operatorname{grad} f$ on its domain, then that f is unique up to an additive constant. For the crucial $\vec{F}$ above, we now show that f doesn't exist, by narrowing down the possible fs to one candidate, and then showing that that candidate doesn't work.

Exercise 9.3.5. *For this exercise, it is helpful to know that $\frac{d}{du}\arctan u = (1 + u^2)^{-1}$. Also, the trigonometric identity $\tan(\theta + \pi/2) = -1/\tan\theta$ implies that $\arctan(x_2/x_1)$ and $\arctan(-x_1/x_2)$ differ by an additive constant, where they are both defined.*

1. *Show that $f = \arctan(x_2/x_1) + C_1$ in the region where $x_1 > 0$.*

2. *Show that* $f = \arctan(-x_1/x_2) + C_2$ *in the region where* $x_2 > 0$.

3. *Show that* $f = \arctan(x_2/x_1) + C_3$ *in the region where* $x_1 < 0$.

4. *Show that* $f = \arctan(-x_1/x_2) + C_4$ *in the region where* $x_2 < 0$.

5. *Prove that there is no way to pick the constants* C_1, C_2, C_3, C_4 *to make* f *well defined on the entire domain of* $\vec{F}$. *(Hint: Compute* f *at the four points* $(\pm 1, \pm 1, 0)$.*)*

We now have an example of $\vec{F} \in \ker \operatorname{curl} - \operatorname{im} \operatorname{grad}$ on $X = \mathbb{R}^3 - Z$, where Z is the curve parametrized by $\vec{x}(t) = (0, 0, t)$. Let's see whether we can get examples $\vec{F}$ on $\mathbb{R}^3 - Z$ for other curves Z.

Exercise 9.3.6. *What if instead* Z *is parametrized by* $\vec{x}(t) = (1, 4, t)$? *Can you find an example* $\vec{F}$ *on* $\mathbb{R}^3 - Z$ *then? What if* Z *is parametrized by* $\vec{x}(t) = (-3, t, 7)$ *or* $\vec{x}(t) = (t, 1, 0)$*? What if* Z *is a union of disjoint curves such as those? Here are some harder questions: What if* Z *is parametrized by* $\vec{x}(t) = (t, t, 3)$ *or* $\vec{x}(t) = (0, t, t^3)$*?*

We haven't shown that this is the only way to produce vector fields $\vec{F} \in \ker \operatorname{curl} - \operatorname{im} \operatorname{grad}$, but yes, this is essentially the only way. Taking this fact for granted...

Exercise 9.3.7. *What is the dimension of* $\ker \operatorname{curl} / \operatorname{im} \operatorname{grad}$*, in terms of topological properties of* X*?*

9.3.5 div **versus** curl

The discrepancy between $\operatorname{im} \operatorname{curl}$ and $\ker \operatorname{div}$ is more difficult to discern. The crucial example is

$$\vec{G} = \frac{\vec{x}}{\|\vec{x}\|^3}$$

(or any scalar multiple of that $\vec{G}$, such as the force fields of Newtonian gravity or Coulomb's law).

Exercise 9.3.8. *On what domain* X *is that* $\vec{G}$ *defined? Check that* $\vec{G} \in \ker \operatorname{div}$ *on that domain.*

To show that $\vec{G}$ is not the curl of any $\vec{F}$, we need to remember (or learn) some more calculus. Let S be a smooth surface in $\mathbb{R}^3$. At any point of S, there are two unit normal vectors. We assume that S is orientable, meaning that there is a way to choose one of these two normals, consistently across all of S, to produce a smooth normal vector field $\vec{N}$ on S.

Let $\vec{G}$ be a vector field on a region of $\mathbb{R}^3$ containing S. The surface integral $\iint_S \vec{G} \cdot d\vec{S}$ is essentially the integral of $\vec{G} \cdot \vec{N}$. If $\vec{G}$ and $\vec{N}$ point in similar directions everywhere on S, then the integral is positive. If they point in opposing directions everywhere on S, then the integral is negative. If S is a water filter in a water flow, and $\vec{G}$ is the velocity field of the flow, then the integral measures the flux of water through the filter.

When a smooth surface has a boundary, as the surface drawn above does, that boundary is always a smooth curve C. Surfaces without boundary effectively have boundary curve $C = \emptyset$. In the special case where $\vec{G}$ is the curl of some vector field $\vec{F}$, Stokes's theorem lets us turn the surface integral of $\vec{G}$ into a line integral of $\vec{F}$ along C:

$$\iint_S \operatorname{curl} \vec{F} \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{x}.$$

Exercise 9.3.9. *Without doing any calculations, argue — maybe without full rigor — that the crucial* $\vec{G}$ *defined at the start of this section is not the curl of any vector field* $\vec{F}$ *on its domain.*

Exercise 9.3.10. *By doing a small modification to the example* $\vec{G}$ *above, can you get vector fields* $\vec{G} \in \ker \operatorname{div} - \operatorname{im} \operatorname{curl}$ *on other domains* X*?*

We haven't shown that this is the only way to produce vector fields $\vec{G} \in \ker \operatorname{div} - \operatorname{im} \operatorname{curl}$, but yes, this is essentially the only way. Taking this fact for granted...

Exercise 9.3.11. *What is the dimension of* $\ker \operatorname{div} / \operatorname{im} \operatorname{curl}$*, in terms of topological properties of* X*?*

9.4 de Rham cohomology

We can't fully explain how these ideas generalize, because there's a lot of theory that needs building. But here's a non-rigorous, impressionistic sketch of some of the key ideas.

Imagine that you're in a smooth n-manifold X. In a little patch of X, you have coordinates $x_1, x_2, \ldots, x_n$. These coordinates lead to a concept of differential 1-forms $dx_1, dx_2, \ldots, dx_n$. They are linear transformations from the tangent spaces of the manifold to $\mathbb{R}$.

As you move from patch to patch, these differential forms patch together in a well behaved way to give differential forms on the whole manifold X. And their behavior on the whole manifold is what makes them topologically interesting. But we're going to continue to write differential forms as if we're working in a coordinate patch, because we don't have time to develop the machinery properly.

There is a product operation, denoted $\wedge$, on these differential forms. A product $dx_i \wedge dx_j$ is a 2-form, a product $dx_i \wedge dx_j \wedge dx_k$ is a 3-form, etc. For any fixed $q \geq 0$, q-forms can be added and scaled, and they form a vector space denoted Ω^q. A q-form $\omega \in \Omega^q$ can be written as

$$\omega = \sum_I f_I \; dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_q},$$

where $I = (i_1, i_2, \ldots, i_q) \in \{1, \ldots, n\}^q$ and $f_I : X \to \mathbb{R}$ is smooth. A 0-form is simply a scalar function $f : X \to \mathbb{R}$.

We can define a differentiation function d, which is called the exterior derivative. It is actually a collection of functions $d_0, d_1, d_2, \ldots$, where $d_q : \Omega^q \to \Omega^{q+1}$ is defined by

$$\begin{aligned} d_q(\omega) &= d_q\left(\sum_I f_I \; dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_q}\right) \\ &= \sum_I d_q\left(f_I \; dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_q}\right) \\ &= \sum_I \sum_{j=1}^n \left(\frac{\partial f_I}{\partial x_j} \; dx_j \wedge dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_q}\right). \end{aligned}$$

Clairaut's theorem implies that $d_{q+1} \circ d_q$ is the zero function. In other words, $\operatorname{im} d_q \subseteq \ker d_{q+1}$. For $q \geq 0$, we define the qth de Rham cohomology to be the quotient vector space

$$H^q(X) = \ker d_{q+1} / \operatorname{im} d_q.$$

It contains rich information about the topological structure of the space X.

In the special case where X is an open subset of $\mathbb{R}^3$, all of this simplifies. We can view differential forms as scalar fields or vector fields, and the exterior derivative d manifests as grad, curl, or div, depending on the situation. The preceding sections give you some idea of the topological structure that de Rham cohomology detects.

9.5 Stokes's theorem

Have you noticed that differential forms resemble the stuff that you traditionally see inside integrals? For example, the 3-form $(x^3 + y^2 - \cos z)\, dx \wedge dy \wedge dz$ resembles the stuff inside the triple integral

$$\int_0^1 \int_3^4 \int_2^3 x^3 + y^2 - \cos z \; dx \; dy \; dz.$$

That's no accident. Differential forms are the mathematical objects that can be integrated on manifolds. We now sketch one of their loveliest theorems.

The table below summarizes five big theorems of calculus. Depending on how much calculus you've studied and where you studied it, you'll know at least one of them but maybe not all of them.

dimension	flat version	curved version
1	$\int_{[a,b]} F'(x)\; dx = F(b) + -F(a).$	$\int_C (\mathrm{grad}\, f) \cdot d\vec{x} = f(\vec{x}(b)) + -f(\vec{x}(a)).$
2	$\iint_D \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\; dx\; dy = \int_C \vec{F} \cdot d\vec{x}.$	$\iint_S \mathrm{curl}\, \vec{F} \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{x}.$
3	$\iiint_E \mathrm{div}\, \vec{G}\; dx\; dy\; dz = \iint_S \vec{G} \cdot d\vec{S}.$	

The upper-left theorem is the fundamental theorem of calculus, where $[a, b]$ is a closed, bounded interval. The upper-right theorem is the fundamental theorem of calculus for line integrals, where C is a curve parametrized by $\vec{x}$. Both the interval $[a, b]$ and the curve C are one-dimensional. Intuitively, $[a, b]$ is the special case of C, where C happens to be straight. Make sense so far?

The middle row of the table features Green's theorem on the left, with D a region of the plane $\mathbb{R}^2$, and Stokes's theorem on the right, with S a surface in $\mathbb{R}^3$. The curve C is the boundary curve of D or S. Both D and S are two-dimensional. Intuitively, D is the special case of S, where S happens to be flat.

The third row of the table features the divergence theorem, also known as Gauss's theorem or Ostrogradsky's theorem. It concerns an integral over a solid three-dimensional region E, and another integral over the surface S that bounds E. The right side of the third row is blank, not because there's no theorem to state, but because the theorem is about curved 3-manifolds, which are rarely taught in calculus courses.

All five equations in the table follow a single pattern. In each equation, the right integrand is a function, and the left integrand is some kind of derivative of that function. In each equation, the left integral is over an n-dimensional space, and the right integral is over its $(n - 1)$-dimensional boundary. (The boundary of a one-dimensional space is a zero-dimensional space, which is a set of discrete points.

Those points are assigned positive and negative signs according to a certain rule. An integral over a set of discrete points is simply a sum.)

Differential forms allow this pattern to be stated elegantly. The boundary of an n-manifold-with-boundary X is an $(n-1)$-manifold ∂X. An $(n-1)$-form ω can be integrated over ∂X, and its exterior derivative $d\omega$ is an n-form, which can be integrated over X. The pattern is then

$$\int_X d\omega = \int_{\partial X} \omega.$$

This is a theorem, which holds in all dimensions. It's called Stokes's theorem, in honor of the original Stokes's theorem, which is the $n = 2$ case.

This generalized Stokes's theorem is a great example of how math often becomes simpler as it becomes more advanced. But this simplification is not free; it comes at a cost of greater abstraction. To put it another way: Math is the study of patterns, and abstraction helps us express patterns among patterns.