9.1 Contraction mapping principle

For the first half of this material, you can consult almost any analysis textbook. For the second half, a good reference is Section 57 of these functional analysis lecture notes by Sigurd Angenent:

`https://people.math.wisc.edu/~angenent/Free-Lecture-Notes/725notes.pdf`

Definition 9.1.1. *Let (X, d) be a metric space. A sequence $x_0, x_1, x_2, \ldots$ of points in X is said to be* ***Cauchy*** *if for all $\epsilon > 0$ there exists an N such that $d(x_n, x_m) < \epsilon$ for all $n, m \geq N$. Moreover, (X, d) is* ***complete*** *if every Cauchy sequence converges.*

Example 9.1.2. *Let $T > 0$. Let $C([0, T], \mathbb{R})$ be the set of all continuous functions $x : [0, T] \to \mathbb{R}$. It is an infinite-dimensional vector space. On it, define the* sup *norm*

$$\|x\| = \sup_{t \in [0,T]} |x(t)| = \max_{t \in [0,T]} |x(t)|.$$

Then $C([0, T], \mathbb{R})$ is a complete metric space under the metric induced by this norm.

Definition 9.1.3. *A function $f : (X, d_X) \to (Y, d_Y)$ is* ***Lipschitz continuous*** *if there exists a constant $K \geq 0$ such that $d_Y(f(x), f(y)) \leq K d_X(x, y)$. Moreover, a Lipschitz continuous $F : (X, d) \to (X, d)$ with $K < 1$ is a* ***contraction****.*

The intuition that you should have is that Lipschitz continuity is stronger than continuity but weaker than continuous differentiability (C^1). My sense of the math culture is: Analysts like having these fine gradations of continuity, because they illuminate exactly what is required to prove theorems, while topologists consider the fine gradations to be too fussy.

Theorem 9.1.4. *A contraction of a complete metric space has a unique fixed point. That is, if (X, d) is complete and $F : (X, d) \to (X, d)$ is a contraction, then there exists a unique $x \in X$ such that $F(x) = x$.*

Proof. Here's a sketch of the key steps to fill in. Starting from any $x_0 \in X$, define $x_{n+1} = F(x_n)$. You get a Cauchy sequence, which converges to some $x \in X$. That's the fixed point. Moreover, it's unique. □

Theorem 9.1.5. *Let $f : \mathbb{R} \to \mathbb{R}$ be Lipschitz continuous with constant L. Let $0 < T < L$. Then the ordinary differential equation $\frac{dx}{dt} = f(x)$ with initial condition $x(0) = x_0$ has a unique solution $x \in C([0, T], \mathbb{R})$.*

Proof. By the fundamental theorem of calculus, the differential equation is tantamount to the integral equation

$$x(t) = x_0 + \int_0^t f(x(s))\, ds.$$

Define $F : C([0, T], \mathbb{R}) \to C([0, T], \mathbb{R})$ by

$$F(x)(t) = x_0 + \int_0^t f(x(s))\, ds.$$

Then $x \in C([0, T], \mathbb{R})$ solves the integral equation if and only if $F(x) = x$. Well, it can be shown that F is a contraction on $C([0, T], \mathbb{R})$ with $K = LT$. So it has a unique fixed point x. □