A.A. A function $f : X_T \to Y_S$ is *continuous* if, for every open set U in Y_S, the preimage $f^{-1}(U)$ is open in X_T.

A.B. If $Y \subseteq X_T$, then the *subspace topology* is the coarsest topology on Y that makes the inclusion $i : Y \hookrightarrow X_T$ continuous.

A.C. If $f : X_T \to Y$ is surjective, then the *quotient topology* is the finest topology on Y that makes f continuous.

A.D. If X_T and Y_S are spaces, then the *product topology* on $X \times Y$ is the coarsest topology that makes the projections $p : X \times Y \to X_T$ and $q : X \times Y \to Y_S$ continuous.

B.A. This X can be covered by two open sets U and V, each of which is homeomorphic to $\mathbb{T}^2$ with a puncture, and such that $U \cap V$ is homeomorphic to $\mathbb{S}^1 \times \mathbb{R}$. [Draw a picture.] So $U \cap V$ is path-connected, and BSVK applies.

The punctured torus U has the homotopy type of a figure eight, so its fundamental group is isomorphic to $\mathbb{Z} * \mathbb{Z}$. That is, it is free on two generators a and b, which are the equator and the meridian of the torus that was punctured. Symmetrically, the fundamental group of V is free on generators c and d, which are the equator and meridian of the other torus.

So BSVK says that $\pi_1(X)$ is generated by the images of a, b, c, d under the inclusions. The images of a and c are loops in X that go "around" the handles, while the images of b and d are loops in X that go "through" the handles. [Draw a picture.] BSVK doesn't tell us about any relations that might hold among these generators. [That's SVK's job.]

B.B. The Abelianization will be generated by the same four elements, but now allowing them to commute with each other. So the Abelianization will be $\mathbb{Z}^4$, possibly with some relations that BSVK doesn't capture.

B.C. The Abelianization of $\pi_1(X)$ is in fact $\mathbb{Z}^4$ with no additional relations. So our answer to B.B was pretty good. We just didn't know that there would be no relations.

C. No, the kernel of grad is strictly larger than the image of zero (when X is non-empty). For starters, the kernel contains all constant functions. In fact, it contains all functions that are constant on each connected component of X — and no other functions. When choosing a function f in the kernel, we can choose f's constant value on each connected component independently of the others. Therefore dimension of the kernel equals the number of connected components of X. In contrast, the dimension of the image of zero is 0.

D.A. We are asking whether there exists an $x \in \mathbb{S}^2$ such that $f(-x) = f(x)$. Yes, the Borsuk-Ulam theorem says so.

D.B. We are asking whether there exists an $x \in \mathbb{S}^2$ such that $-f(x) = f(x)$. In other words, is 0 in the image of f? We do not know this. Some maps f satisfy this requirement (such as $f(x, y, z) = (x, y)$), and other maps f do not (such as $f(x, y, z) = (1, 1)$).

E.A. The compact connected surfaces are the sphere $\mathbb{S}^2$, the connected sum $\mathbb{T}^2 \# \cdots \# \mathbb{T}^2$ of $g \geq 1$ tori, and the connected sum $\mathbb{RP}^2 \# \cdots \# \mathbb{RP}^2$ of $g \geq 1$ real projective planes.

E.B. All of the surfaces listed in E.A have non-isomorphic Abelianized fundamental groups: the trivial group for $\mathbb{S}^2$, $\mathbb{Z}^{2g}$ for the connected sum of g tori, and $\mathbb{Z}^{g-1} \times \mathbb{Z}/2\mathbb{Z}$ for the connected sum of g real projective planes. Therefore these surfaces have non-isomorphic fundamental groups. Therefore they are not homeomorphic.

[Some students gave a different criterion: That the surfaces all differ by either Euler characteristic or orientability (or both). That's true, but we have not studied orientability. In fact, we haven't even established the preliminaries needed to define it, and Munkres doesn't either. And Euler characteristic is not sufficient on its own. So this answer did not earn full credit.]