Please write your name at the top of this page and nowhere else.

In addition to this cover page, there should be problems A–E spread over six pages.

You are allowed one sheet of paper with notes on both sides, as we have discussed. Otherwise, no notes, books, calculators, computers, etc. are allowed.

If a problem is unclear and you cannot obtain clarification, then write your interpretation of the problem, so that I can evaluate your solution relative to your interpretation. You might be penalized, if your interpretation makes the problem much easier than it should be. Certainly you should never interpret a problem in a way that renders it trivial.

Except where otherwise noted, all problems require explanation. Incorrect answers with good work often earn partial credit. Correct answers without justification rarely earn full credit. Write as if your audience is a typical classmate — not a professor. In doing so, you (hopefully) show enough detail, that I can evaluate whether you understand your arguments.

Except where otherwise directed, you may cite material (definitions, theorems, etc.) that we have defined or proved in class, in the assigned textbook readings, or in the assigned homework. You do not need to re-define or re-prove any of that material. You may not cite other material without developing it first. When in doubt, ask me.

Pictures often help both you and your reader.

You have 150 minutes (2.5 hours). Good luck. :)

In the following four definitions, fill in all of the blanks. Explanation is NOT required, but add explanation if you feel that your answers need it.

A.A. A function $f : X_T \to Y_S$ is *continuous* if $\boxed{}$.

A.B. If $Y \subseteq X_T$, then the *subspace topology* is the $\boxed{}$ topology on Y that $\boxed{}$.

A.C. If $f : X_T \to Y$ is $\boxed{}$, then the *quotient topology* is the $\boxed{}$ topology on Y that $\boxed{}$.

A.D. If X_T and Y_S are spaces, then the *product topology* on $X \times Y$ is the $\boxed{}$ topology that $\boxed{}$.

We have proved this "BSVK" theorem: Let $X = U \cup V$, where U and V are open in X and $U \cap V$ is path-connected. Let $i : U \hookrightarrow X$ and $j : V \hookrightarrow X$ be the inclusions. Fix a base point $x \in U \cap V$. Then the images of $i_* : \pi_1(U) \to \pi_1(X)$ and $j_* : \pi_1(V) \to \pi_1(X)$ generate $\pi_1(X)$.

In Problem B, we focus on the case where $X = \mathbb{T}^2 \# \mathbb{T}^2$ is the connected sum of two tori.

B.A. What does BSVK tell you about $\pi_1(X)$? Explain as much as you can — for example, the isomorphism types of any of the groups involved, if you know them — even if you can't reach a simple final answer.

B.B. Based on your answer to B.A, what can you say about the Abelianization of $\pi_1(X)$?

B.C. What is the Abelianization of $\pi_1(X)$? In other words, how good is your answer to B.B?

Let X be an open set in $\mathbb{R}^3$. Let $\mathbf{S} = \{\text{smooth } f : X \to \mathbb{R}\}$ and $\mathbf{V} = \{\text{smooth } \vec{F} : X \to \mathbb{R}^3\}$ be the vector spaces of smooth scalar and vector fields on X, respectively. Recall that grad : $\mathbf{S} \to \mathbf{V}$ is defined by $\operatorname{grad} f = \langle \partial f/\partial x_1, \partial f/\partial x_2, \partial f/\partial x_3 \rangle$. Let $\mathbf{0}$ be the vector space consisting of just the number 0. Define zero : $\mathbf{0} \to \mathbf{S}$ by saying that zero(0) is the function $X \to \mathbb{R}$ defined by $x \mapsto 0$. (Warning: This setup is intentionally slightly different from what we did in class.)

C. Is the kernel of grad equal to the image of zero? Explain thoroughly and precisely, based on the topology of X.

Let $\mathbb{S}^{n-1} \subseteq \mathbb{R}^n$ be the unit sphere. Let $a : \mathbb{R}^n \to \mathbb{R}^n$ be given by $a(x) = -x$. Then the restriction of a to $\mathbb{S}^{n-1}$ gives a map $\mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$, which we also call a. Let $f : \mathbb{S}^2 \to \mathbb{R}^2$ be a map.

D.A. Does there exist an $x \in \mathbb{S}^2$ such that $(f \circ a)(x) = f(x)$?

D.B. Does there exist an $x \in \mathbb{S}^2$ such that $(a \circ f)(x) = f(x)$?

E.A. List the compact connected surfaces (up to homeomorphism).

E.B. How do we know that the surfaces listed in E.A are not homeomorphic to each other?