B.A. First, p is continuous and surjective. Now, let $b = (r, \cos t, \sin t)$ be an arbitrary point in the base space $\mathbb{R}_{>0} \times \mathbb{S}^1$. Let U be the interval $(r/2, 3r/2)$, so that U is a neighborhood of r in $\mathbb{R}_{>0}$. Let V be the arc of $\mathbb{S}^1$ that runs from $(\cos(t-1), \sin(t-1))$ to $(\cos(t+1), \sin(t+1))$, so that V is a neighborhood of $(\cos t, \sin t)$ in $\mathbb{S}^1$. Then $U \times V$ is a neighborhood of b. Its pre-image $p^{-1}(U \times V)$ is

$$\cdots \;\cup\; U \times (t-1-2\pi, t+1-2\pi) \;\cup\; U \times (t-1, t+1) \;\cup\; U \times (t-1+2\pi, t+1+2\pi) \;\cup\; \cdots.$$

The map p sends each of those terms homeomorphically onto $U \times V$. So $U \times V$ is evenly covered, and p is a covering map.

B.B. Yes, we can use this covering map to calculate the fundamental group of the base space. Let $b = (1, 1, 0) \in \mathbb{R}_{>0} \times \mathbb{S}^1$. The covering space $\mathbb{R}_{>0} \times \mathbb{R}$ is a convex subset of $\mathbb{R}^2$ and hence is simply connected. So the lifting correspondence

$$\phi : \pi_1(\mathbb{R}_{>0} \times \mathbb{S}^1, b) \to p^{-1}(b) = \{1\} \times 2\pi\mathbb{Z}$$

is a bijection. Identify the codomain $\{1\} \times 2\pi\mathbb{Z}$ with the group $\mathbb{Z}$ via the bijection $(1, 2\pi k) \mapsto k$. Then ϕ is a homomorphism, by an argument similar to the one that we used for the covering map $\mathbb{R} \to \mathbb{S}^1$. We conclude that the fundamental group of $\mathbb{R}_{>0} \times \mathbb{S}^1$ is isomorphic to $\mathbb{Z}$. [If homotopy equivalence were part of this exam, then we could use it to confirm this result.]

[Problem C was assigned as homework, but I didn't realize that fact until just a few minutes before handing out the exam, which was too late to change anything. So this problem became a have-you-done-your-homework check, which is okay.]

C.A. Let $F : X \times I \to X$ be a homotopy between the constant map $X \to \{x_0\}$ and the identity $X \to X$. So $F(x, 0) = x_0$ and $F(x, 1) = x$ for all $x \in X$. For any map $f : Y \to X$, let $G : Y \times I \to X$ be given by $G(y, t) = F(f(y), t)$. Then G is continuous, $G(y, 0) = F(f(y), 0) = x_0$, and $G(y, 1) = F(f(y), 1) = f(y)$. Thus G is a homotopy between f and the constant map $Y \to \{x_0\}$. So f is homotopic to that constant map, and similarly g is homotopic to that constant map, and hence f is homotopic to g.

C.B. Let $x \in X$. Let F be the homotopy from part A of the problem. Define $\alpha : I \to X$ by $\alpha(t) = F(x, t)$. Then $\alpha(0) = F(x, 0) = x_0$ and $\alpha(1) = F(x, 1) = x$. So α is a path from x_0 to x. Because every $x \in X$ is connected to x_0 in this way, X is path-connected.

[In Problem D, what we're calling "multiplicity" is actually called *degree*. I gave it a different name in an attempt to make cheating more difficult.]

D.A. The identity map is $h \circ h$, where h is the first coordinate reflection, which has multiplicity

-1 by the first property. Therefore the identity map has multiplicity $-1 \cdot -1 = 1$, by the second property.

D.B. Although it's a small abuse of notation, let's conflate the matrix B_s with the linear transformation $\mathbb{R}^3 \to \mathbb{R}^3$ that it describes. Because this linear transformation is a rotation, it carries $\mathbb{S}^2$ to $\mathbb{S}^2$. So we have a map $B_s : \mathbb{S}^2 \to \mathbb{S}^2$. This map is homotopic to the identity map $\mathbb{S}^2 \to \mathbb{S}^2$ by the homotopy $H(\vec{x}, t) = B_{ts}(\vec{x})$. Thus the multiplicity of B_s is 1, by the third property. Now the third coordinate reflection can be written as $B_{\pi/2} \circ h \circ B_{-\pi/2}$, where h is the first coordinate reflection. Shall we check?

$$\begin{aligned}
B_{\pi/2}(h(B_{-\pi/2}(\vec{x}))) &= B_{\pi/2}\left(h\left(\begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)\right) \\
&= B_{\pi/2}\left(h\left(\begin{bmatrix} -z \\ y \\ x \end{bmatrix}\right)\right) \\
&= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}\begin{bmatrix} z \\ y \\ x \end{bmatrix} \\
&= \begin{bmatrix} x \\ y \\ -z \end{bmatrix}.
\end{aligned}$$

Thus the multiplicity of the third coordinate reflection is $1 \cdot -1 \cdot 1 = -1$, by the second property.

Similarly, the second coordinate reflection can be written as $C_{\pi/2} \circ h \circ C_{-\pi/2}$, and it has degree $1 \cdot -1 \cdot 1 = -1$ too.

D.C. The antipodal map is the composition of the three coordinate reflections, so it has multiplicity $-1 \cdot -1 \cdot -1 = -1$.

D.D. I'll prove the contrapositive. Suppose that h has no fixed point. Consider $F : \mathbb{S}^2 \times I \to \mathbb{R}^3$ defined by $F(x, t) = h(x) + t(a(x) - h(x))$. This F is the straight-line homotopy between h and the antipodal map a. It misses the origin, because the line segment between $h(x)$ and $a(x)$ does not contain the origin, because $h(x) \neq x$. So we can define $G : \mathbb{S}^2 \times I \to \mathbb{S}^2$ by $G(x, t) = F(x, t)/\|F(x, t)\|$, and this G is a homotopy between h and a. By the third property, $\text{mult}(h) = \text{mult}(a) = -1$.