**A.A**. We want $f$ to be continuous. Making $T$ fine — in the extreme case, discrete — makes it easier for $f$ to be continuous. So we want $T$ to be the coarsest topology that makes $f$ continuous.

**A.B**. A subset $U \subseteq X$ is open if and only if there exists an open $V \subseteq Y_S$ such that $U = f^{-1}(V)$. Because $f$ is injective, an equivalent criterion is: A subset $U \subseteq X$ is open if and only if there exists an open $V \subseteq Y_S$ such that $f(U) = V \cap f(X)$. [I don't see any simpler way to say it, because of what comes next...]

**A.C**. This construction is extremely similar to the subspace topology. In fact, the subspace topology is the special case where $X \subseteq Y$ and $f$ is the inclusion $i : X \hookrightarrow Y$. Here's another way to say it. An injection $f : X \to Y_S$ is a bijection onto its image. Let $g : X \to f(X)$ be this bijection, and let $i : f(X) \hookrightarrow Y$ be the inclusion, so that $f = i \circ g$. Give $f(X)$ its subspace topology from $Y_S$. Then declare a topology on $X$ by declaring that $g$ is a homeomorphism. I claim that this topology is $T$.

**B**. [This problem is not easy for a timed exam. On the other hand, it was similar to an assigned homework problem, and it was a recommended exercise from Section 24. I omit the picture, but you should not. I give two arguments.]

Shorter argument: If $f(0) = 0$ or $f(1) = 1$, then we are done. So suppose that $f(0) > 0$ and $f(1) < 1$. Let $g : [0, 1] \to \mathbb{R}$ be defined by $g(x) = f(x) - x$. Then $g$ is continuous, $g(0) > 0$, and $g(1) < 0$. By the intermediate value theorem, there exists a $c \in (0, 1)$ such that $g(c) = 0$. But then $f(c) = c$, and $f$ has a fixed point.

Longer argument: Suppose, for the sake of contradiction, that $f(x) \neq x$ for all $x$. Then $f(0) > 0$, $f(1) < 1$, and the graph of $f$ does not intersect the diagonal

$$D = \{(x, x) : x \in [0, 1]\} \subseteq [0, 1] \times [0, 1].$$

Define $g : [0, 1] \to [0, 1] \times [0, 1]$ by $g(x) = (x, f(x))$. I claim [proof omitted] that $g$ is continuous. Its image is the graph of $f$, which does not intersect $D$. So $g$ maps into the subspace $S = [0, 1] \times [0, 1] - D$, and is continuous as a function into $S$. But $S$ is not connected. One connected component, $S_0$, contains $g(0)$, and the other, $S_1$, contains $g(1)$. I claim that this separation of $S$ induces a separation $g^{-1}(S_0), g^{-1}(S_1)$ of $[0, 1]$. The argument is similar to our proof that path connectedness implies connectedness. But $[0, 1]$ is connected. This contradiction implies that there exists an $x$ such that $f(x) = x$.

**C.A**. First, I claim that $Q$ is at least as fine as the product topology. The basis for the product topology consists of sets of the form $U \times V$, where $U$ and $V$ are open in $X_T$ and $Y_S$ respectively. The set $U \times V$ projects to $U \subseteq X_T$ and $V \subseteq Y_S$, which are open. Therefore such a $U \times V$ is an element of the sub-basis that induces $Q$. It follows that every set open under the product topology is also open under $Q$. In other words, $Q$ is at least as fine as the product topology.

Second, I claim that $Q$ is finer than the product topology. Consider

$$U = B((2,0),1) \cup B((0,2),1) \cup \{(2,2)\} \subseteq \mathbb{R}^2.$$

This set $U$ is not open under the product topology, but it is open under $Q$, because it's a sub-basis element, because its projections are $(1,3)$ and $(1,3)$.

[Another example is the diagonal $U = \{(x,x) : x \in \mathbb{R}\}$. Another example is $U = \mathbb{R}^2 - B(0,1)$. In fact, after I published these solutions, some students pointed out that $Q$ is the discrete topology. To see so, fix a point $(x,y) \in \mathbb{R}^2$. Then the line of slope 1 through $(x,y)$ is open under $Q$, and the line of slope $-1$ is open under $Q$, so their intersection $\{(x,y)\}$ is open too. By this argument, every one-point subset of $(X \times Y)_Q$ is open, and $Q$ is the discrete topology.]

**C.B**. The product topology is the coarsest topology that makes $p$ and $q$ continuous. Any finer topology, such as $Q$, also makes them continuous. So yes, we can immediately conclude that $p$ and $q$ are continuous. [If your answer to C.A is “finer” or “equal”, then your answer to C.B should be “yes”. If your answer to C.A is “coarser” or “not comparable”, then your answer to C.B should be “no”.]

**D.A**. Fix $x \in X_T$. For any other $y \in X_T$, let $U_y, V_y$ be the open sets guaranteed by the $T_1$ condition. Then

$$X - \{x\} = \bigcup_{y \neq x} V_y,$$

which is open. Thus $\{x\}$ is closed.

**D.B**. Let $x \neq y \in X_T$. Let $U = X - \{y\}$ and $V = X - \{x\}$. Because every one-point set is closed, we know that $U$ and $V$ are open. And $x \in U \not\ni y$ and $x \notin V \ni y$. Thus $X_T$ is $T_1$.