There is one multi-part exercise today. You will prove that the connected sum of a torus and a real projective plane is homeomorphic to the connected sum of three real projective planes:

$$\mathbb{T}^2 \# \mathbb{RP}^2 \cong \mathbb{RP}^2 \# \mathbb{RP}^2 \# \mathbb{RP}^2.$$

The entire exercise is a series of manipulations of labeled polygonal regions. Some people find this kind of thing fun. Remember that you are allowed to systematically rename labels, flip regions over, etc.

A.A. Prove that $\mathbb{T}^2 \# \mathbb{RP}^2$ is equivalent to $ccaba^{-1}b^{-1}$. (By "equivalent", I mean that $\mathbb{T}^2 \# \mathbb{RP}^2$ is homeomorphic to the space that you get by gluing a hexagonal region with labeling $(c, c, a, b, a^{-1}, b^{-1})$. Will you need to rename labels? Maybe.)

A.B. Prove that that space is equivalent to $abcbac$. (To do this, I cut the hexagon into two quadrilaterals and then reglued in a certain way. If you don't know what I mean by cutting, then see Figure 76.1. On the new edges that you make by cutting, you need to introduce a new label. If you don't know what I mean, then see the top of Figure 76.3.)

A.C. Prove that that space is equivalent to $bbac^{-1}ac$. (Hint: To do this, I cut the hexagon into a triangle and a pentagon and then reglued.)

A.D. Prove that that space is equivalent to $aabbcc$ (which is the connected sum of three real projective planes).

In general, it can be proven that the connected sum of g tori and one real projective plane is homeomorphic to the connected sum of $2g + 1$ real projective planes. That is, summing with a real projective plane converts tori into real projective planes at a one-to-two exchange rate. You have just proved the $g = 1$ case. The general proof is carried out in Section 77. If you look carefully at the proof of Lemma 77.4, you can see the entirety of Problem A buried inside there.

You might be tempted to study Section 77 in detail, and then use that machinery to crank out Problem A. I won't stop you, but you're certainly not saving yourself any work. The $g = 1$ case is more understandable than the general machinery.