There are four problems labeled A–D. The big theme is rotations, which are important throughout theoretical math, applied math, physics, engineering, computer graphics, etc.

Let $\mathbb{R}^{n\times n}$ be the set of real $n \times n$ matrices. If you start with an $n \times n$ matrix, and list its entries column-by-column (or row-by-row, or in any other consistent order), then you obtain an (n^2)-dimensional vector. In this way, $\mathbb{R}^{n\times n}$ is bijectively identified with $\mathbb{R}^{n^2}$. Moreover, the Frobenius inner product on $\mathbb{R}^{n\times n}$ (Day 05 Homework) is identified with the dot product on $\mathbb{R}^{n^2}$. So the induced norm $\|\ldots\|$ on $\mathbb{R}^{n\times n}$ matches the standard norm on $\mathbb{R}^{n^2}$, and the induced metrics and topologies match too. Everything is pretty simple.

Let $\mathrm{SO}(n) \subseteq \mathbb{R}^{n\times n}$ be the set of matrices R such that $R^\top R = I$ and $\det R = 1$. Under the usual identification of matrices with linear transformations (with respect to the standard basis of $\mathbb{R}^n$), $\mathrm{SO}(n)$ is the set of rotations of $\mathbb{R}^n$.

A.A. Prove that $\mathrm{SO}(n)$ is a closed subset of $\mathbb{R}^{n\times n}$. (This is a medium-length exercise. Hint: Show that each of the $n^2 + 1$ scalar equations, that define $\mathrm{SO}(n)$, defines a closed subset.)

A.B. Prove that, for all $R \in \mathrm{SO}(n)$, $\|R\| \leq \sqrt{n}$. (This is a short exercise.)

A.C. Prove that $\mathrm{SO}(n)$ is compact. (This is a short exercise.)

By the way, $\mathrm{SO}(n)$ is a manifold of dimension $n(n-1)/2$. And it's a group. It's a very nice kind of mathematical object, which is called a compact *Lie group*. And it's path-connected.

Because we seem to live in $\mathbb{R}^3$, $\mathrm{SO}(3)$ is the $\mathrm{SO}(n)$ that shows up most often in scientific applications: spacecraft maneuvers, robotic control, computer vision, motions of tectonic plates across Earth's surface, medical imaging, etc. A peculiar feature of $\mathrm{SO}(3)$ is its relationship to another structure called the *quaternions*. There's a lot to say about them, but I'm going to tell you just what you need, to do the problems below.

A quaternion is a vector $\mathbf{a} = (a_0, a_1, a_2, a_3) \in \mathbb{R}^4$. As you'd expect, the norm of $\mathbf{a}$ is $\|\mathbf{a}\| = \sqrt{a_0^2 + a_1^2 + a_2^2 + a_3^2}$, and a unit quaternion is a quaternion of norm 1. Commonly we regard the last three components of $\mathbf{a}$ as a vector $\vec{a} = (a_1, a_2, a_3) \in \mathbb{R}^3$, and we write the quaternion $\mathbf{a}$ in the somewhat weird notation $a_0 + \vec{a}$. For any unit quaternion $\mathbf{a} = a_0 + \vec{a}$ and any $\vec{v} \in \mathbb{R}^3$, let

$$R_{\mathbf{a}}(\vec{v}) = -2(\vec{a} \times \vec{v}) \times \vec{a} + 2a_0(\vec{a} \times \vec{v}) + \vec{v}.$$

One can prove that $R_{\mathbf{a}}(\vec{v})$ is the vector $\vec{v}$ rotated about some axis through some angle, which are encoded into $\mathbf{a}$ in some way. Moreover, every rotation of $\mathbb{R}^3$ can be written as $R_{\mathbf{a}}$ for some unit quaternion $\mathbf{a}$. In this way, the association $\mathbf{a} \mapsto R_{\mathbf{a}}$ gives a surjection from the space of unit quaternions onto SO(3). Moreover, this surjection is a two-to-one covering map. You will now check a little part of this last claim.

B. Prove that $R_{\mathbf{a}} = R_{-\mathbf{a}}$. That is, $\mathbf{a} = (a_0, a_1, a_2, a_3)$ and $-\mathbf{a} = (-a_0, -a_1, -a_2, -a_3)$ map to the same rotation of $\mathbb{R}^3$. (This is a short exercise.)

Finally, Problems C and D give you some payoff for learning this quaternion material. You may assume all of the facts stated above.

C. What is the fundamental group of SO(3)? (This is a short exercise.)

D. By mimicking the Day 22 homework, describe a system of plotting data, in which a data set of n rotations of $\mathbb{R}^3$ appears as a set of n points in a three-dimensional plot. What is the shape of that plot? Are there any caveats, that you would want to tell a user of that plot? (This is a short exercise. I do not expect a lot of rigor.)