A. Section 26 Exercise 4, which is about compact subspaces of metric spaces.

For the remainder of this homework, we need the concept of partition of unity, which we start developing now. Define $f_0 : \mathbb{R} \to \mathbb{R}$ by

$$f_0(x) = \begin{cases} 1 + x & \text{if } -1 \leq x \leq 0, \\ 1 - x & \text{if } 0 \leq x \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

For all $k \in \mathbb{Z}$, define $f_k : \mathbb{R} \to \mathbb{R}$ by $f_k(x) = f_0(x - k)$. Then $\{f_k\}_{k \in \mathbb{Z}}$ is a family of continuous functions such that, at each x, only finitely many $f_k(x)$ are non-zero and the non-zero $f_k(x)$ sum to 1. Read the definitions of *support* and *partition of unity* in Section 36.

B. Use the family $\{f_k\}$ above to construct a partition of unity on the interval $[-2, 2]$. Explicitly state the open sets U_k and the supports. Be precise. Include a picture.

Now we need another tool. Let $\mathbb{S}^n \subseteq \mathbb{R}^{n+1}$ be the unit sphere. The *stereographic projection* from the point $(0, 0, \ldots, 0, 1) \in \mathbb{S}^n$ is the function $f : \mathbb{S}^n - \{(0, 0, \ldots, 0, 1)\} \to \mathbb{R}^n$ defined by

$$f(x_1, x_2, \ldots, x_n, x_{n+1}) = \left(\frac{x_1}{1 - x_{n+1}}, \frac{x_2}{1 - x_{n+1}}, \ldots, \frac{x_n}{1 - x_{n+1}} \right).$$

Stereographic projection is one of the most famous functions in mathematics. It's been known for thousands of years. It has applications in cartography, geology, chemistry, complex analysis, etc. It has beautiful properties, most of which we don't need right now. All we need right now is that f is a homeomorphism. You may henceforth assume so.

C.A. Modify the f above to get a homeomorphism $g : \mathbb{S}^n - \{(0, 0, \ldots, 0, -1)\} \to \mathbb{R}^n$.

Together, f and g are enough to demonstrate that $\mathbb{S}^n$ is a n-manifold. Make sure that you understand this statement, but I don't need to see any documentation. For the rest of this problem, we restrict to the $n = 1$ case.

C.B. Let $U, V \subseteq \mathbb{S}^1$ be the domains of f and g respectively. Construct a partition of unity dominated by $\{U, V\}$. You can describe it using either algebraic expressions or precise pictures.

C.C. Following the proof of Theorem 36.2 and using your answer to C.B, construct an embedding of $\mathbb{S}^1$ into some $\mathbb{R}^N$. In particular, what value of N do you get?