

This first problem is optional. Your assigned homework is problems A, B, C below. This week in class I claimed that a norm on a vector space V induces a metric on the set V by $d(\vec{v}, \vec{w}) = \|\vec{v} - \vec{w}\|$.

Optional: Prove this claim.

Near the start of Section 21, Munkres claims: If (X, d) is a metric space and $Y \subseteq X$, then the restriction of d to $Y \times Y$ is a metric on Y , and the topology that it induces on Y matches the subspace topology on Y .

A. Prove the second part of this claim, that the topologies match. (You may assume the first part of the claim, that the metric restricts to a metric, which is easier to prove.)

For all positive-definite, symmetric $n \times n$ matrices G , we know that $\langle \vec{v}, \vec{w} \rangle_G = \vec{v}^\top G \vec{w}$ is an inner product on \mathbb{R}^n and $\|\vec{v}\|_G = \sqrt{\langle \vec{v}, \vec{v} \rangle_G}$ is the norm induced by that inner product. We also know, from previous homework, that

$$\forall \epsilon > 0 \quad \exists \delta > 0 \quad \forall \vec{v} \in \mathbb{R}^n \quad \|\vec{v}\|_I < \delta \Rightarrow \|\vec{v}\|_G < \epsilon$$

and

$$\forall \epsilon > 0 \quad \exists \delta > 0 \quad \forall \vec{v} \in \mathbb{R}^n \quad \|\vec{v}\|_G < \delta \Rightarrow \|\vec{v}\|_I < \epsilon.$$

B. All values of G induce the same topology on \mathbb{R}^n . Prove so. (Hint: Lemma 13.3.)

Epilogue: Here's a concrete example, using $\vec{v} = (1, 0)$ and $\vec{w} = (0, 1)$ in \mathbb{R}^2 . For $G = I$, \vec{v} and \vec{w} have norm 1 and make an angle of $\pi/2$. But for

$$G = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix},$$

\vec{v} and \vec{w} have norm $\sqrt{2}$ and make an angle of $2\pi/3$. So we see that differing values of G produce different geometries. But our work above shows that all G produce the same topology. This is a solid example of how topology is “geometry with lots of detail ignored”.

C.A. Let X_T be a space with the discrete topology. Prove that X_T is metrizable.

C.B. Let X_T be a metrizable space with only finitely many points. Prove that T is the discrete topology on X .