

**B.** Suppose, for the sake of contradiction, that  $R$  is a recognizer for  $A$ . Let  $P$  be a Turing machine that, on input  $\langle M, w \rangle$ , does this:

1. Build a Turing machine  $N$  that accepts all inputs.
2. Run  $R$  on  $\langle M, N, w \rangle$ , and output whatever  $R$  outputs.

Then  $P$  accepts  $\langle M, w \rangle$  if and only if  $R$  accepts  $\langle M, N, w \rangle$ , which happens if and only if  $M$  does not accept  $w$ , which is true if and only if  $\langle M, w \rangle \in ACC_{TM}^c$ . So  $P$  is a recognizer for  $ACC_{TM}^c$ . But that language is unrecognizable. This contradiction implies that  $A$  is also not recognizable.

**C.** The criterion can be restated as “ $L(M)$  does not contain any strings of length less than 5”. So it depends not on the implementation of  $M$  but only on  $L(M)$ . This language  $A$  is what we call a property of recognizable languages. Moreover, it’s non-trivial, because some  $\langle M \rangle$  are in the language (e.g.,  $M$  rejects all inputs) and some  $\langle M \rangle$  aren’t (e.g.,  $M$  accepts all inputs). So the desired result is an immediate consequence of Rice’s theorem.

**D.** Assume for the sake of contradiction that  $A$  is context-free. Let  $p$  be a pumping length for  $A$ . Let

$$s = (00)^p(11)^p01(00)^p(11)^p.$$

Then  $s \in A$  and  $|s| = 8p + 2 \geq p$ , so  $s$  can be pumped in the usual way. I see four cases of interest, for where  $vxy$  can be, in  $s = uvxyz$ .

1. If  $vxy$  is a substring of the first  $(00)^p(11)^p$ , then pumping  $uvxyz$  to  $uv^2xy^2z$  gives a string in which the first part (before the 01 divider) is longer than the second part, so this string is not in  $A$ .

2. If  $vxy$  is a substring of the second  $(00)^p(11)^p$ , then pumping  $uvxyz$  to  $uv^2xy^2z$  gives a string in which the second part is longer than the second part, so this string is not in  $A$ .

3. If  $vy$  contains any part of the 01 divider in the middle of the string, then there are several subcases. In each subcase, pumping  $uvxyz$  to  $uv^2xy^2z$  damages or repeats the divider, so that the pumped string is not in  $A$ . (Per the prompt, this case’s argument leaves out some detail.)

4. Otherwise,  $v$  is a substring of the first  $(11)^p$  and  $y$  is a substring of the second  $(00)^p$ . Then pumping  $uvxyz$  to  $uv^2xy^2z$  gives a string in which the first part mismatches the second part, so this string is not in  $A$ .

In all four cases,  $s$  can be pumped to leave  $A$ , in contradiction of the pumping lemma. This contradiction shows that  $A$  is not context-free.

**E.** I use a two-tape Turing machine. Upon startup, it puts a beginning-of-tape marker such as  $\vdash$  on the second tape.

Then it starts scanning the input, loading all  $a$ s onto the second tape. As soon as it finds a  $b$  in the input, it starts matching the  $b$ s on the first tape (still scanning rightward) with the  $a$ s on

the second tape (scanning leftward). As soon as it finds a  $c$  in the input, it starts matching the  $cs$  on the first tape (still scanning rightward) with the  $as$  on the second tape (scanning rightward). As soon as it finds a  $d$  in the input, it starts matching the  $ds$  on the first tape (still scanning rightward) with the  $as$  on the second tape (scanning leftward).

In the end, the two-tape Turing machine accepts if the numbers of  $as$ ,  $bs$ ,  $cs$ , and  $ds$  all agree, with no other errors (such as finding an  $a$  after a  $c$ ) along the way. Otherwise, it rejects.