

A.A. We compute

$$\frac{dx}{du} = \begin{bmatrix} -r \sin u \\ r \cos u \\ 0 \\ 0 \end{bmatrix}, \quad \frac{dx}{dv} = \begin{bmatrix} 0 \\ 0 \\ -r \sin v \\ r \cos v \end{bmatrix}$$

and

$$g = \begin{bmatrix} \frac{dx}{du} \cdot \frac{dx}{du} & \frac{dx}{du} \cdot \frac{dx}{dv} \\ \frac{dx}{du} \cdot \frac{dx}{dv} & \frac{dx}{dv} \cdot \frac{dx}{dv} \end{bmatrix} = \begin{bmatrix} r^2 & 0 \\ 0 & r^2 \end{bmatrix} = r^2 I.$$

A.B. The point $p = (r, 0, r, 0)$ corresponds to $(u, v) = (0, 0)$. At that point,

$$\frac{dx}{du} = \begin{bmatrix} 0 \\ r \\ 0 \\ 0 \end{bmatrix}, \quad \frac{dx}{dv} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ r \end{bmatrix}, \quad X = \begin{bmatrix} 0 \\ 3 \\ 0 \\ 4 \end{bmatrix} = \frac{3}{r} \frac{dx}{du} + \frac{4}{r} \frac{dx}{dv}.$$

Therefore X is in the span of the coordinate vectors, and so $X \in T_p S$.

A.C. We just learned that the coordinates of X in the coordinate vector basis are $(3/r, 4/r)$.

Similarly, the coordinates of Y are $(4/r, 3/r)$. Therefore

$$X \cdot Y = \begin{bmatrix} 3/r & 4/r \end{bmatrix} \begin{bmatrix} r^2 & 0 \\ 0 & r^2 \end{bmatrix} \begin{bmatrix} 4/r \\ 3/r \end{bmatrix} = \begin{bmatrix} 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \end{bmatrix} = 3 \cdot 4 + 4 \cdot 3 = 24.$$

[That result agrees with $X \cdot Y = 24$ computed using the ordinary dot product in $\mathbb{R}^4$.]

B. If $\dot{B}$ is constant, then

$$\begin{aligned} 0 &= \ddot{B} \\ &= \frac{d}{ds}(\tau N) \\ &= \dot{\tau} N + \tau \dot{N} \\ &= \dot{\tau} N + \tau(-\kappa T - \tau B) \\ &= -\kappa \tau T + \dot{\tau} N - \tau^2 B. \end{aligned}$$

But T, N, B are linearly independent. So the only way a linear combination of them equals 0 is when the coefficients are all 0. The third coefficient tells us that $\tau = 0$, which forces the first and second coefficients to be 0 as well. So α is a regular curve with $\kappa > 0$ and $\tau = 0$. By a result proved in homework, the image of α must lie in a plane, and T and N must lie in that plane,

and B must be one of the two unit vectors perpendicular to that plane. Moreover, B cannot continuously flip from one option to the other. So B is constant.

[Bonus question: The fact that the proof above uses only the third coefficient suggests that it is too heavy-handed. Suppose that the problem didn't require use of the Frenet-Serret equations. How could you then prove the result using a shorter, simpler geometric argument?]

C.A. [I omit the drawing, although you should not.] An example in $\mathbb{R}^2$ would look like the letter α . That is, the regular curve would smoothly cross itself, and at the crossing point it would fail to be a curve.

C.B. A subset C of $\mathbb{R}^n$ is a *curve* if for every point $p \in C$ there exists an open $I \subseteq \mathbb{R}$, a smooth $\alpha : I \rightarrow \mathbb{R}^n$, an open $W \subseteq \mathbb{R}^n$, and a smooth $F : W \rightarrow \mathbb{R}$ such that $p \in \alpha(I) = W \cap C$ and $F(\alpha(u)) = u$ for all $u \in I$.

D.A. True. If the acceleration a is a scalar multiple of the velocity v , then the perpendicular component of acceleration

$$a^\perp = a - a^\parallel = a - \text{proj}_v a = a - \frac{a \cdot v}{v \cdot v} v$$

is 0, so $\kappa = |a^\perp| / |v|^2 = 0$.

D.B. True. We have proved that the eigenvalues of g are positive. Therefore g does not have 0 as an eigenvalue. Therefore g is invertible. Alternatively, g is symmetric, so we can use the spectral theorem on it. We find that g diagonalizes as $g = QDQ^\top$, where Q is orthogonal, D is diagonal, and the diagonal entries of D are positive. Then D^{-1} has the reciprocal diagonal entries, and $g^{-1} = QD^{-1}Q^\top$.

D.C. False. We also need the gradient ∇f to be non-zero everywhere on S . For example, the level set $f(x) = x_1x_2 = 0$ is not a surface.