

There are three required problems and one optional problem.

I claim that $f : \mathbb{S}^2 \rightarrow \mathbb{R}$ given by $f(x) = |x|^{-2}$ is smooth (as a function on a surface).

Problem A.A. Verify that f is smooth using the definition of smoothness for such functions.

Problem A.B. Verify that f is smooth using the lemma about extending f to $\tilde{f}$ on an open set W . In fact, give me two extensions: one where $W = \mathbb{R}^3$, and one where $W = \mathbb{R}^3 \setminus \{(0, 0, 0)\}$.

Let $x : (0, \pi) \times (-\pi, \pi) \rightarrow \mathbb{R}^3$ be our usual spherical coordinates parametrization given by $x(u, v) = (\sin u \cos v, \sin u \sin v, \cos u)$. Let S be the image of x . Let $f : S \rightarrow \mathbb{R}$ be given by $f(x) = x_1^2 + x_2 x_3$. Let $p = x(q) \in S$.

Problem B.A. Compute df_p using its definition. That is, if $X = X_{(1)} \frac{dx}{du}(q) + X_{(2)} \frac{dx}{dv}(q) \in T_p S$, then tell me what $df_p(X)$ is, in terms of $X_{(1)}$ and $X_{(2)}$.

Problem B.B. Compute df_p by extending f to a function $g : \mathbb{R}^3 \rightarrow \mathbb{R}$, computing the 1×3 matrix $\partial g / \partial x$, and computing $\partial g / \partial x \cdot X$.

[This next problem is Problem 4.5 with slight changes, mostly in notation. It's a glimpse of the start of Morse theory, which is an important tool in geometry and topology.] Let $S \subseteq \mathbb{R}^n$ be a surface, and let $w \in \mathbb{R}^n$ be a unit vector. Define $f : S \rightarrow \mathbb{R}$ by $f(x) = w \cdot x$.

Problem C. Show that f is a smooth function on S and that the derivative df_p is the zero linear transformation if and only if w is perpendicular to S at p .

The following problem is optional.

Problem D. Problem 4.6 from our textbook.