

This Day 10 homework is due on Day 13 (Monday) because of our exam on Day 11 (Wednesday). The next homework assignment, assigned on Day 12 and due on Day 14, resumes our regular homework schedule.

There are three required problems and one optional problem.

Problem A.A. Suppose that $dx_q : \mathbb{R}^2 \rightarrow T_p S$ preserves angles. That is, for all non-zero $w, y \in \mathbb{R}^2$, the angle between w and y in $\mathbb{R}^2$ equals the angle between $df_q(w)$ and $df_q(y)$ in $\mathbb{R}^n$. What condition(s) must $g(q)$ obey, then? (Hint: Evaluate $w^\top g y$ for two of your favorite orthogonal vectors w, y , to learn something about the off-diagonal entries of g . Then evaluate $w^\top g y$ for another orthogonal pair w, y , to learn something about the on-diagonal entries of g .)

Problem A.B. Once you have an answer to Problem A.A, prove the converse: If $g(q) \dots$, then dx_q preserves angles.

In general, we showed that the area distortion factor of a local parametrization x is $\sqrt{\det g}$. Let's say that x "preserves areas at q " if $\sqrt{\det g(q)} = 1$.

For this problem specifically, we're using the x that is quasi-inverse to the stereographic projection from the north pole of $\mathbb{S}^2$ onto the equatorial plane.

Problem B.A. Does dx_q preserve angles when $q = (0, 0)$? Does dx_q preserve angles at points q on the unit circle in $\mathbb{R}^2$? Does dx_q preserve angles as q goes to infinity — that is, when q goes far from $(0, 0)$ in any direction?

Problem B.B. Does x preserve areas at $q = (0, 0)$? On the unit circle? As q goes to infinity?

The unit sphere $\mathbb{S}^2 \subseteq \mathbb{R}^3$ is circumscribed by a unit cylinder whose axis is parallel to the x_3 -axis. (Draw a picture?) Let S be the part of the unit sphere that's parametrized by our usual spherical coordinates:

$$S = \mathbb{S}^2 \setminus \{x : x_1 \leq 0, x_2 = 0\}.$$

Let T be this part of the circumscribing cylinder:

$$T = \{x : x_1^2 + x_2^2 = 1, -1 < x_3 < 1\} \setminus \{x : x_1 \leq 0, x_2 = 0\}.$$

Let $P : T \rightarrow S$ be projection along horizontal lines toward the x_3 -axis. That is, if $r \in T$, then there is a unique line parallel to the x_1 - x_2 -plane that passes through r and the x_3 -axis. This line intersects S in a single point between r and the x_3 -axis. Let $P(r)$ be that intersection point.

Problem C. Choosing U carefully, explicitly construct an $x : U \rightarrow S$ that is a local parametrization at every $p \in S$. This x must first send $q \in U$ to a point in T and then use P to send that point to S . If you do this well, then x preserves area at every $q \in U$. Prove so.

Epilogue: This x and its inverse, and the projection P and its inverse, together constitute a pretty famous example, which has been known for about 2,300 years.

The following problem is optional.

Problem D. Let S and T be surfaces in $\mathbb{R}^3$. Let $p \in S \cap T$, and let x and y be local parametrizations of S and T , respectively, about p . How would you define the angle between S and T at p ? And how would you compute it? Does your computation happen in $\mathbb{R}^3$ or in $\mathbb{R}^2$? Can you do it in the other one?