

There are three required problems and no optional problems.

At the start of class today, we proved the following lemma (which is Lemma 1 of Section 3.1, with superficial differences in presentation).

Lemma: Let $x : U \rightarrow \mathbb{R}^n$ be a local parametrization of a surface S . Let $\alpha : I \rightarrow \mathbb{R}^n$ be smooth such that $\alpha(I) \subseteq x(U)$. Then there exists a smooth $\beta : I \rightarrow U$ such that $\alpha(t) = x(\beta(t))$ for all $t \in I$.

I claim that changing the first sentence of the lemma to “Let $x : U \rightarrow \mathbb{R}^n$ be a regular surface” breaks the lemma. That is, the lemma is no longer true. In this two-step problem, you construct a counter-example. The first step is supposed to capture the intuition, and the second step is supposed to convince us that the intuition could be made precise.

Problem A.A. Draw an example, from an earlier homework, of a regular surface that is not a surface. In that drawing, draw an example of an α that breaks the lemma, and explain why it breaks the lemma (non-rigorously, possibly using additional pictures).

Problem A.B. Suppose that we have a regular surface $x : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ such that these two strips are both contained in the image of x :

$$\begin{aligned} (-1, 1) \times \{0\} \times \mathbb{R} &= \{x : -1 < x_1 < 1, x_2 = 0\}, \\ \{0\} \times (-1, 1) \times \mathbb{R} &= \{x : x_1 = 0, -1 < x_2 < 1\}. \end{aligned}$$

Explicitly define an α that breaks the lemma. (You are not required to rigorously prove that it breaks the lemma.)

The tangent space $T_p S$ is a vector subspace of $\mathbb{R}^n$, meaning that it is closed under addition and scalar multiplication, meaning that for all $X, Y \in T_p S$ and $c \in \mathbb{R}$,

$$X + Y \in T_p S, \quad cX \in T_p S.$$

Problem B.A. My friend Gary is trying to prove that $T_p S$ is closed under addition. His claimed proof is quoted below. What’s wrong with it?

“Let $X, Y \in T_p S$. So there exist curves α, β in S such that $\alpha(0) = p$, $\dot{\alpha}(0) = X$, $\beta(0) = p$, and $\dot{\beta}(0) = Y$. Define a curve γ (on the intersection of the domains of α and β) by $\gamma(t) = \alpha(t) + \beta(t) - p$. Then $\gamma(0) = p$ and $\dot{\gamma}(0) = X + Y$. So $X + Y \in T_p S$.”

Problem B.B. Prove that $T_p S$ is closed under scalar multiplication. Like Gary’s attempted proof, you should work from the definition of $T_p S$ as the set of velocities of curves.

For the last problem, you need three ingredients.

The first ingredient is the definition of the hyperbolic sine function: $\sinh x = (e^x - e^{-x})/2$.

The second ingredient is spherical coordinates, which we used today in class. Recall that $x : (0, \pi) \times (-\pi, \pi) \rightarrow \mathbb{R}^3$ given by

$$x(u, v) = (\sin u \cos v, \sin u \sin v, \cos u)$$

parametrizes (almost all of) the unit sphere $\mathbb{S}^2$. Suppose you want to integrate a function $f(x)$ over $\mathbb{S}^2$. You might think that

$$\iint_{\mathbb{S}^2} f(x) \, dA = \int_{-\pi}^{\pi} \int_0^{\pi} f(x(u, v)) \, du \, dv.$$

But that's incorrect! There is a “distortion factor” between areas on the sphere and areas in the domain of x . This area distortion is a recurring theme in our study of surfaces. The area distortion factor for this particular x is $\sin u$, and the corrected equation is

$$\iint_{\mathbb{S}^2} f(x) \, dA = \int_{-\pi}^{\pi} \int_0^{\pi} f(x(u, v)) \sin u \, du \, dv.$$

(From Calculus 3, you might have memorized a volume distortion factor of $\rho^2 \sin \phi$ for spherical coordinates. Well, here we lock $\rho = 1$, and we use u for ϕ , so we get $\rho^2 \sin \phi = \sin u$.)

Here's the third ingredient. If you take a course in probability (or statistics), you will find yourself working with random variables (or data) that are real numbers — in other words, points in $\mathbb{R}$. In more advanced courses, you may work with multivariate random variables, which are points in $\mathbb{R}^n$. In even more advanced study, there is a geometric kind of probability, where random variables can be points in curves such as $\mathbb{S}^1$, in surfaces such as $\mathbb{S}^2$, or in higher-dimensional curved spaces such as

$$\text{SO}(p) = \{Q \in \mathbb{R}^{p \times p} : QQ^\top = I, \det Q = 1\}.$$

This geometric probability theory finds application in robotics, astronomy, biomechanics, geology, etc. One of the most popular probability distributions in this theory is defined as follows. Fix a constant $\mu \in \mathbb{S}^2$ and a constant $k > 0$. Define the probability density function $f : \mathbb{S}^2 \rightarrow \mathbb{R}$ by

$$f(x) = ce^{k\mu \cdot x}.$$

Here, c is a constant that depends on k . It must be chosen so that the integral of f over $\mathbb{S}^2$ is 1.

Problem C. For simplicity, assume that $\mu = (0, 0, 1)$. Prove that

$$c = \frac{k}{4\pi \sinh k}.$$