

There are three required problems and two optional problems.

In class we discussed the stereographic projection  $F$  from the north pole of  $\mathbb{S}^2$  onto the equatorial plane, and the local parametrization  $x$  of  $\mathbb{S}^2$  that is quasi-inverse to  $F$ .

Problem A. Altering that example as little as possible, define stereographic projection  $F$  from the south pole onto the equatorial plane, and the quasi-inverse local parametrization  $x$  corresponding to that  $F$ .

Given a smooth curve  $\alpha : I \rightarrow \mathbb{R}^2$ , we can “extrude”  $\alpha$  into the third dimension to make a smooth  $x : I \times \mathbb{R} \rightarrow \mathbb{R}^3$  defined by

$$x(u, v) = (\alpha_1(u), \alpha_2(u), v).$$

When all goes well, the image of  $x$  is a surface called a *cylinder*. The special case where  $\alpha$  is (most of) a circle is the most popular example, but in general  $\alpha$  can be quite different from a circle. For example,  $\alpha(u) = (u, \sin u)$  makes  $x$  a “wavy curtain”, and that’s a cylinder.

Problem B.A. Under what assumptions on  $\alpha$  is  $x$  a regular surface? Prove your answer. (My proof is pretty short.)

Problem B.B. Under what assumptions on  $\alpha$  is the image of  $x$  a surface? You are not expected to prove your answer.

Recall the example from class, in which  $M$  was a symmetric  $3 \times 3$  matrix (not necessarily positive-definite) and  $S = \{x : f(x) = x^\top Mx = 1\} \subseteq \mathbb{R}^3$ . We proved that  $\nabla f = 0$  nowhere on  $S$ . So, as long as  $S$  is non-empty,  $S$  is a surface. Let’s finish that example now.

Problem C.A. Describe  $S$  in the special case where  $M$  is diagonal with diagonal entries  $0, 0, 1$ . And in the case where  $M$  is diagonal with diagonal entries  $0, 1, 1$ . And in the case where  $M$  is diagonal with diagonal entries  $1, 1, 1$ .

Problem C.B. Prove that  $S$  is non-empty if  $M$  has at least one positive eigenvalue. (This problem is more of a linear algebra problem than a differential geometry problem, but practicing linear algebra is a fantastic idea.)

The following problems are optional.

Problem D. Problem 2.3b in our book. (Hint: Review basic facts of cosh and sinh?)

Problem E. Problem 2.4 in our book. (Warning: The constants  $a, b$  have the opposite relationship from the one used in class.)