

This Day 03 homework should be seriously attempted before Day 04 (Monday) and handed in at the start of class on Day 05 (Wednesday). There are four required problems and two optional problems.

Problem A. Let $\alpha : \mathbb{R} \rightarrow \mathbb{R}^3$ be regular. Because we're in $\mathbb{R}^3$, we have access to a special tool: the cross product $\times$. Working from the $\kappa = |\mathbf{a}^\perp| / |\mathbf{v}|^2$ definition given in class, prove that

$$\kappa = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3}.$$

(Hint: The dot product has an algebraic definition ($\mathbf{v} \cdot \mathbf{w} = \sum v_i w_i$) and a geometric interpretation ($\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}| |\mathbf{w}| \cos \text{angle}(\mathbf{v}, \mathbf{w})$). Similarly, the cross product has an algebraic definition and a geometric interpretation. Use the geometric interpretation. Look it up, if you need to. A picture of that geometric interpretation is an important part of my solution to this problem.)

Problem B. Let $\alpha : \mathbb{R} \rightarrow \mathbb{R}^2$ be regular. Let x and y be its two component functions, so that $\alpha(u) = (x(u), y(u))$. Prove that

$$\kappa = \frac{|\dot{x}\ddot{y} - \ddot{x}\dot{y}|}{(\dot{x}^2 + \dot{y}^2)^{3/2}}.$$

(Update 2026/9/21: The numerator needs to be in an absolute value sign. Sorry for omitting it earlier.) (Hint: Is there a way to use Problem A?)

Problem C. Prove that $d\mathbf{T}/ds = \kappa\mathbf{N}$, as was claimed in class. (The point of this exercise is to verify that our book's treatment is logically equivalent to our class's treatment. So you are not allowed to cite the book. The work here is difficult if you don't know what we did in class, and much easier if you do know.)

Let $\alpha : I \rightarrow \mathbb{R}^n$ be regular. Let $u_0 \in I$ be a time at which $\kappa(u_0) \neq 0$. I am trying to construct the circle, parametrized by a function $\beta : \mathbb{R} \rightarrow \mathbb{R}^n$, that best approximates the trace of α at $\alpha(u_0)$. Specifically, this β should satisfy:

1. The point $\beta(0) \in \mathbb{R}^n$ equals the point $\alpha(u_0) \in \mathbb{R}^n$.
2. The two curves have the same unit tangent vector $\mathbf{T}$ at that point.
3. The two curves have the same unit normal vector $\mathbf{N}$ at that point.
4. The two curves have the same curvature κ at that point.

Problem D. Find an explicit parametrization $\beta(v)$ in terms of the relevant data for α , namely $\alpha(u_0)$, $\kappa(u_0)$, $\mathbf{T}(u_0)$, and $\mathbf{N}(u_0)$. (Hint: What is the radius of the circle? What is the center? So what is the parametrization? Pictures strongly help!)

The two problems below are optional extra practice. See your textbook for more practice.

Problem E. Find the curvature of the catenary curve $\alpha(u) = (u, \cosh u)$. (Here, $\cosh$ is a hyperbolic trig function. It helps to know that $\cosh'(x) = \sinh(x)$, $\sinh'(x) = \cosh(x)$, $\cosh^2 x - \sinh^2 x = 1$, and $\cosh x \geq 1$. Also, this is part of Problem 1.6 in our textbook.)

Let $\alpha : I \rightarrow \mathbb{R}^n$ be a regular curve with velocity $\mathbf{v} : I \rightarrow \mathbb{R}^n$ as usual. Define $g : I \rightarrow (0, \infty)$ by $g(u) = \mathbf{v}(u) \cdot \mathbf{v}(u)$. Now I'm going to ask you a pretty weird question. Its importance becomes apparent later in the course.

Problem F. Suppose that you know everything about g — for example, you can compute $g'(u)$, $g''(u)$, etc. From this knowledge of g , can you compute the curvature κ ?