

To clarify: This Day 02 homework should be seriously attempted before Day 03 (Friday), and it should be handed in on paper at the start of class on Day 04 (Monday). There are three required problems (A, B, C) and two optional problems (D, E).

Also, please respond to my questionnaire e-mail today (Day 02, Wednesday)!

Problem A.A. Define $\alpha : \mathbb{R} \rightarrow \mathbb{R}^2$ by

$$\alpha(u) = (u, u^2) = \begin{bmatrix} u \\ u^2 \end{bmatrix}.$$

Sketch the image of α (which our textbook calls the *trace*). And is this α smooth? And is it regular? And compute the unit tangent vector field $\mathbf{t}$, or explain why it's not possible.

Problem A.B. Repeat the same tasks for $\alpha(u) = (u^3, u^6)$.

For constants $a > b > 0$, define $\alpha : \mathbb{R} \rightarrow \mathbb{R}^2$ by

$$\alpha(u) = (a \cos u, b \sin u) = \begin{bmatrix} a \cos u \\ b \sin u \end{bmatrix}.$$

This α parametrizes an ellipse with long axis of length $2a$ and short axis of length $2b$.

Problem B.A. At which points on the ellipse is the speed of α maximized and minimized? (Hint: Optimize the square of the speed, rather than the speed itself. Why is that equivalent?)

Earth's orbit around the Sun is an ellipse. For a suitable choice of time units, length units, constants a and b , and coordinate system, this ellipse can be parametrized by the α above. The Sun doesn't sit at the origin — it sits at one of the foci of the ellipse — but that's okay.

Problem B.B. Does Problem B.A tell us where the Earth is moving fastest and slowest relative to the Sun? (You are not expected to use any outside knowledge of planetary motion.)

For positive constants c and λ , define $\alpha : \mathbb{R} \rightarrow \mathbb{R}^2$ by

$$\alpha(u) = ce^{\lambda u}(\cos u, \sin u) = ce^{\lambda u} \begin{bmatrix} \cos u \\ \sin u \end{bmatrix} = \begin{bmatrix} ce^{\lambda u} \cos u \\ ce^{\lambda u} \sin u \end{bmatrix}.$$

The image is a logarithmic spiral, like the shape of a nautilus shell. It exhibits a self-similarity property, a key aspect of which we explore now.

Problem C.A. Compute the angle between $\alpha(u)$ and $\dot{\alpha}(u)$ at all times u . (Hint: The answer is a constant. I want you to compute that constant.)

Problem C.B. Explicitly re-parametrize the logarithmic spiral in terms of arc length s relative to $u_0 = 0$. (This problem does not use Problem C.A, but some of the computational steps are shared.)

Want more practice? See below for some optional problems. You are not required to hand them in, they will not be graded, and they don't earn any extra credit, but they might help you learn. Of course, the textbook also has problems to try.

Let $r > 0$ and $c > 0$ be constant. The function $\alpha : \mathbb{R} \rightarrow \mathbb{R}^3$ defined by

$$\alpha(u) = (r \cos u, r \sin u, cu) = \begin{bmatrix} r \cos u \\ r \sin u \\ cu \end{bmatrix}$$

is a regular curve. It's a helix of radius r and "rising speed" c .

Problem D. Re-parametrize this curve by arc length relative to $u_0 = 0$.

Recall from the first day of class that $\mathbb{S}^{n-1} = \{\mathbf{x} : |\mathbf{x}| = 1\} \subseteq \mathbb{R}^n$ is the unit sphere in $\mathbb{R}^n$. Suppose that $\alpha : \mathbb{R} \rightarrow \mathbb{S}^{n-1}$ is a regular curve. (That is, α is a regular curve in $\mathbb{R}^n$, whose image just happens to be contained in $\mathbb{S}^{n-1}$.)

Problem E. Find a relationship between α and $\dot{\alpha}$. (This might seem like an impossibly open-ended question, but you have very little to go on, so it's actually quite contained.)