

Today's homework consists of three tasks:

Read the course web site (<http://www.joshuadavis.us/teaching/2026f344/index.html>).

Do the four-part calculus problem below.

Do the two-part linear algebra problem below.

Because this is our first homework assignment, let's clarify the schedule. It is assigned on Day 01 (Monday). You are expected to seriously attempt it before Day 02 (Wednesday). You are expected to submit your solutions at the start of class on Day 03 (Friday). Why attempt the problems before Day 02, if they're not due until Day 03? Because then you will better understand class on Day 02, and you might have time to meet with me, before the problems are due.

Your solutions should be listed in the order assigned. They should be written in grammatical English sentences with correct use of mathematical jargon and notation. Frequently you need to work out a calculation/proof on scratch paper, then recopy it for your final version. Write "Day 01" or "Day 1" at the top of the front page. If you're handing in multiple pages, then please staple (not paperclip) them.

For the calculus problem, let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function and $k \geq 0$ an integer. We denote the k th derivative of f as $f^{(k)}$ (if it exists). For example, $f^{(0)} = f$ and $f^{(1)} = f'$. We say that f is of class C^k if its first $k + 1$ derivatives

$$f^{(0)}, f^{(1)}, \dots, f^{(k)}$$

all exist and are continuous. For example, f is of class C^0 if and only if f is continuous. We say that f is of class C^∞ or *smooth* if f is of class C^k for all $k \geq 0$. We say that f is of class C^ω or *analytic* if its Taylor-Maclaurin series (the power series constructed by expanding f at 0) converges to f on all of $\mathbb{R}$. These classes of functions nest like this:

$$C^0 \supseteq C^1 \supseteq C^2 \supseteq C^3 \supseteq \dots \supseteq C^\infty \supseteq C^\omega.$$

In this problem, we prove that all of these set containments are proper; in other words, no two of the classes are equal. There are four steps.

Problem A.A: Consider the example

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ x^3 & \text{if } x > 0. \end{cases}$$

Sketch graphs of f and its first few derivatives. And what is the (smallest) class of f ?

Problem A.B: By modifying the preceding example, construct an f that is of class C^k and not of class C^{k+1} , for each $k \geq 0$. You do not need to prove your answer.

Problem A.C: Consider the example

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ e^{-x^{-1}} & \text{if } x > 0. \end{cases}$$

Argue that this f is smooth. (The only worrisome point is $x = 0$. Compute the first few derivatives there, until the pattern becomes clear. You may assume without proof the fact that

$$\lim_{x \rightarrow 0^+} \frac{e^{-x^{-1}}}{x^k} = 0$$

for all positive integers k .)

Problem A.D: Prove that the preceding smooth f is not analytic. (Hint: What would its Taylor-Maclaurin series be, if f were analytic?)

For the linear algebra problem, let G be a positive-definite, symmetric $m \times m$ matrix. (All matrices in this problem, and probably all matrices in this course, are real.)

Problem B.A: Using a theorem mentioned in class today, prove that there exists a symmetric matrix $\sqrt{G}$ such that $(\sqrt{G})^2 = G$. (Update: On the next page, I list some of the linear algebra facts that I used in solving this problem.)

Problem B.B: Prove that there exists an invertible linear transformation $f : \mathbb{R}^m \rightarrow \mathbb{R}^m$ such that

$$v \cdot_G w = f(v) \cdot f(w)$$

for all $v, w \in \mathbb{R}^m$.

Epilogue: So we see that the deformed dot product is identical to the ordinary dot product, up to some “renaming” of vectors by f . This is why the deformed dot product doesn’t produce a deep, interesting theory — as long as G is constant.

Here are some of the definitions and theorems from linear algebra that I used in solving Problem B.A. All matrices are $m \times m$, and all vectors are in $\mathbb{R}^m$.

An *eigenvector* of G is a non-zero $\mathbf{v}$ such that there exists $\lambda \in \mathbb{R}$ such that $G\mathbf{v} = \lambda\mathbf{v}$. In that case, λ is the *eigenvalue* of G associated to the eigenvector $\mathbf{v}$. If $\mathbf{v}$ is an eigenvector with eigenvalue λ , then so is $c\mathbf{v}$ for all non-zero scalars c .

If two or more vectors are orthogonal, then they are linearly independent.

If G has m linearly independent eigenvectors, then it diagonalizes as $G = PDP^{-1}$, where the columns of P are the eigenvectors, D is diagonal, and the diagonal entries of D are the corresponding eigenvalues.

The transpose “distributes reversing order”: for example, $(ABC)^\top = C^\top B^\top A^\top$, and $(\mathbf{v}^\top B \mathbf{w})^\top = \mathbf{w}^\top B^\top \mathbf{v}$.