This document teaches you about the dot product and cross product, including their geometric meanings and how they facilitate backface culling. It also describes spherical coordinates.

1 Dot product

Suppose that you have two d-dimensional vectors $\vec{v}$ and $\vec{w}$, rendered in the same coordinate system as

$$\vec{v} = \begin{bmatrix} v_0 \\ \vdots \\ v_{d-1} \end{bmatrix}, \quad \vec{w} = \begin{bmatrix} w_0 \\ \vdots \\ w_{d-1} \end{bmatrix}.$$

The *dot product* of $\vec{v}$ and $\vec{w}$ is the number $\vec{v} \cdot \vec{w}$ defined as

$$\vec{v} \cdot \vec{w} = \sum_{k=0}^{d-1} v_k w_k = v_0 w_0 + \cdots + v_{d-1} w_{d-1}.$$

The *magnitude* or *length* $||\vec{v}||$ of a vector $\vec{v}$ is defined as the square root of the dot product of the vector with itself:

$$||\vec{v}|| = \sqrt{\vec{v} \cdot \vec{v}} = \sqrt{{v_0}^2 + \cdots + {v_{d-1}}^2}.$$

(This definition might remind you of the Pythagorean theorem. In modern mathematics, the Pythagorean "theorem" is a definition, not a theorem.)

The vectors of length 1 are especially important for describing directions in space. They are called *unit* vectors. If $\vec{v}$ has non-zero length, then dividing $\vec{v}$ by its length produces a unit vector $\frac{1}{||\vec{v}||}\vec{v}$ in the same direction as $\vec{v}$.

The dot product can be described in another way, that reveals its geometric meaning. If you place $\vec{v}$ and $\vec{w}$ tail-to-tail, then there is some angle θ between them. It's a non-obvious fact that

$$\vec{v} \cdot \vec{w} = ||\vec{v}|| \, ||\vec{w}|| \cos \theta.$$

The two vectors $\vec{v}$ and $\vec{w}$ are perpendicular if and only if $\theta = \pi/2 = 90°$, which occurs if and only if $\vec{v} \cdot \vec{w} = 0$. If $\vec{v} \cdot \vec{w} > 0$, then the two vectors point roughly in the same direction. If $\vec{v} \cdot \vec{w} < 0$, then the two vectors point roughly in opposite directions.

Here are some algebraic rules that the dot product obeys. For all vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ and all numbers c,

- $(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$ (linearity in the first argument),
- $(c\vec{u}) \cdot \vec{v} = c(\vec{u} \cdot \vec{v})$ (linearity in the first argument),
- $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$ (linearity in the second argument),
- $\vec{u} \cdot (c\vec{v}) = c(\vec{u} \cdot \vec{v})$ (linearity in the second argument),

- $\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$ (symmetry),
- $\vec{v} \cdot \vec{v} > 0$ unless $\vec{v} = \vec{0}$ (positive definiteness).

From these basic rules, many other properties follow. For example,

$$||\vec{v} - \vec{w}||^2 = (\vec{v} - \vec{w}) \cdot (\vec{v} - \vec{w}) = \vec{v} \cdot \vec{v} - \vec{v} \cdot \vec{w} - \vec{w} \cdot \vec{v} + \vec{w} \cdot \vec{w} = ||\vec{v}||^2 - 2\vec{v} \cdot \vec{w} + ||\vec{w}||^2.$$

2 Cross product

The following concept exists for vectors only in dimensions $d = 0, 1, 3, 7$, and we will discuss it only in dimension $d = 3$. The *cross product* of two 3-dimensional vectors $\vec{v}$ and $\vec{w}$ is defined to be the vector

$$\vec{v} \times \vec{w} = \begin{bmatrix} v_0 \\ v_1 \\ v_2 \end{bmatrix} \times \begin{bmatrix} w_0 \\ w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} v_1 w_2 - v_2 w_1 \\ v_2 w_0 - v_0 w_2 \\ v_0 w_1 - v_1 w_0 \end{bmatrix}.$$

Alternatively, there is a geometric definition of the cross product, in terms of its direction and magnitude. Let θ be the angle between $\vec{v}$ and $\vec{w}$. Then the magnitude of the cross product is

$$||\vec{v} \times \vec{w}|| = ||\vec{v}|| \, ||\vec{w}|| \sin \theta.$$

The direction of the cross product is perpendicular to both $\vec{v}$ and $\vec{w}$. That's not quite enough to complete the definition, because there are two opposite directions that are both perpendicular to $\vec{v}$ and $\vec{w}$. To complete the definition, we specify that the cross product is right-handed. To understand what this means, hold your right hand in the air, with your index finger perpendicular to your thumb, and your middle finger perpendicular to both your thumb and index finger. Suppose that your index finger is $\vec{v}$ and your middle finger is $\vec{w}$. Then your thumb points in the direction of $\vec{v} \times \vec{w}$.

Here are some algebraic rules that the cross product obeys. For all $\vec{u}$, $\vec{v}$, and $\vec{w}$ and all c,

- $(\vec{u} + \vec{v}) \times \vec{w} = \vec{u} \times \vec{w} + \vec{v} \times \vec{w}$ (linearity in the first argument),
- $(c\vec{u}) \times \vec{v} = c(\vec{u} \times \vec{v})$ (linearity in the first argument),
- $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$ (linearity in the second argument),
- $\vec{u} \times (c\vec{v}) = c(\vec{u} \times \vec{v})$ (linearity in the second argument),
- $\vec{v} \cdot \vec{w} = -\vec{w} \cdot \vec{v}$ (anti-symmetry),
- $\vec{u} \times (\vec{v} \times \vec{w}) + \vec{v} \times (\vec{w} \times \vec{u}) + \vec{w} \times (\vec{u} \times \vec{v}) = \vec{0}$ (Jacobi identity).

The cross product is not associative; the Jacobi identity describes how it fails to be associative. The cross product satisfies many more rules than the ones listed here, including several rules where it interacts with the dot product.

3 Backface culling

Here is an example of how the dot and cross products interact in computer graphics. Suppose that we have a closed surface in 3D, that is made out of 3D triangles. Each triangle is a list $\vec{a}$, $\vec{b}$, $\vec{c}$ of 3D points, in counter-clockwise order when viewed from "outside" the surface. We want to determine whether a given triangle is facing toward us or not. If it is facing toward us, then we should render it. If it is facing away from us, then we should not render it, because it is guaranteed to be occluded by some triangles facing toward us, because the surface is closed. So is it facing toward us or not?

The vector $\vec{b} - \vec{a}$ points along one side of the triangle, and the vector $\vec{c} - \vec{a}$ points along another side. The cross product of these two vectors is

$$(\vec{b} - \vec{a}) \times (\vec{c} - \vec{a}) = \begin{bmatrix} (b_1 - a_1)(c_2 - a_2) - (b_2 - a_2)(c_1 - a_1) \\ (b_2 - a_2)(c_0 - a_0) - (b_0 - a_0)(c_2 - a_2) \\ (b_0 - a_0)(c_1 - a_1) - (b_1 - a_1)(c_0 - a_0) \end{bmatrix}.$$

The cross product is perpendicular to the triangle. If it is facing roughly toward the camera, then we should render the triangle. If it is facing roughly away from the camera, then we should not render the triangle. It is implicit, in the way that we've placed coordinates on the screen, that the vector

$$\vec{e} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

points toward the camera. So backface culling boils down to the sign of the dot product

$$\vec{e} \cdot ((\vec{b} - \vec{a}) \times (\vec{c} - \vec{a})) = (b_0 - a_0)(c_1 - a_1) - (b_1 - a_1)(c_0 - a_0).$$

A happy coincidence is that this crucial quantity $(b_0 - a_0)(c_1 - a_1) - (b_1 - a_1)(c_0 - a_0)$ is already computed in our `triRender` function. It is the determinant of the 2×2 matrix

$$\begin{bmatrix} b_0 - a_0 & c_0 - a_0 \\ b_1 - a_1 & c_1 - a_1 \end{bmatrix},$$

which we invert while computing interpolations. So, if this determinant is positive, then we must render the triangle. If it is negative, then we can ignore the triangle. If it is zero, which happens rarely, then the line of sight is parallel to the triangle, so the triangle will not fill any pixels, so we ignore it then too.

4 Spherical coordinates

This section has nothing to do with the dot or cross product. It merely describes a convenient way of making 3D vectors of length 1. Let ρ, ϕ, and θ be three numbers, such that $\rho \geq 0$,

$0 \le \phi \le \pi$, and $-\pi \le \theta \le \pi$. Then

$$\vec{v} = \begin{bmatrix} \rho \sin \phi \cos \theta \\ \rho \sin \phi \sin \theta \\ \rho \cos \phi \end{bmatrix}$$

is a vector of length ρ, whose direction is specified by ϕ and θ. Frequently, but not always, we set $\rho = 1$, so that we produce a vector of length 1.

In geography, each point on the Earth's surface can be specified by a latitude and longitude. Latitude is 90° at the North Pole, 0° at the Equator, and -90° at the South Pole. For example, Northfield is at about 44° latitude, so it's about halfway between the Equator and the North Pole. If latitude is an angle measuring north-south-ness, then longitude is an angle measuring east-west-ness. Longitude is 0° at the Prime Meridian (a great circle that runs through the United Kingdom, Ghana, and several countries between them). Longitude is positive east of there and negative west of there. Northfield is roughly at longitude -93°, and the International Date Line is roughly at longitude $\pm 180^\circ$.

So how does this geography lesson relate to spherical coordinates? Well, imagine the Earth placed in a Cartesian coordinate system. The origin is at the center of the Earth. The x-axis punctures the Earth's surface off the coast of West Africa, where the Equator meets the Prime Meridian. The y-axis punctures the Earth's surface at a certain point on the Equator in the eastern Indian Ocean. The z-axis punctures the Earth's surface at the North Pole. With these conventions in place, longitude is θ, and latitude is $90^\circ - \phi$. Sometimes ϕ is called *co-latitude*, meaning the angle that you add to latitude to get 90°.

Warning: In physics textbooks, the notation is often reversed, so that $0 \le \theta \le \pi$ is the co-latitude and $-\pi \le \phi \le \pi$ is the longitude. There's nothing wrong or right about that notation. You just have to pick a notation and use it consistently.