This tutorial covers a few more points about 3×3 matrices. It focuses on rotations, which are much more complicated in three dimensions than in two dimensions.

1 Inversion and Transposition

In our 2×2 matrix tutorial, we learned about determinants and inverses of 2×2 matrices M. In fact, we learned how to compute determinants and inverses explicitly (because we needed them for our linear interpolator).

The same ideas apply to any 3×3 matrix M. There is a number called the determinant of M, and M has an inverse M^{-1} if and only if $\det M \neq 0$. But we're not going to learn how to compute $\det M$ or M^{-1} in general, because we don't need them in this course. We need only one extremely special case: rotations.

Rotations of three-dimensional space can be represented by 3×3 matrices. They turn out to be drastically more complicated than rotations of two-dimensional space. The following sections give two useful ways to construct three-dimensional rotations. For the rest of this section, suppose that we have already constructed a 3×3 rotation matrix R.

What is $\det R$? It is always 1. And therefore R^{-1} exists.

What is R^{-1}? Well, define the *transpose* of a 3×3 matrix M to be the 3×3 matrix $M^\top$ obtained by reflecting M across its diagonal:

$$M^\top = \begin{bmatrix} M_{00} & M_{01} & M_{02} \\ M_{10} & M_{11} & M_{12} \\ M_{20} & M_{21} & M_{22} \end{bmatrix}^\top = \begin{bmatrix} M_{00} & M_{10} & M_{20} \\ M_{01} & M_{11} & M_{21} \\ M_{02} & M_{12} & M_{22} \end{bmatrix}.$$

In the special case where M is a rotation matrix R, then $R^{-1} = R^\top$. That's great news. It means that, for rotation matrices, inversion is easy, fast, and numerically precise.

2 Describing Rotations In Terms Of Pairs Of Vectors

Suppose that we have two length-1, three-dimensional vectors $\vec{u}$ and $\vec{v}$ that are perpendicular to each other. We also have two length-1, three-dimensional vectors $\vec{a}$ and $\vec{b}$ that are perpendicular to each other. There is a unique rotation of three-dimensional space that transforms $\vec{u}$ to $\vec{a}$ and $\vec{v}$ to $\vec{b}$. We wish to find the 3×3 matrix R that describes that rotation.

We compute the cross product $\vec{w} = \vec{u} \times \vec{v}$ and form the 3×3 matrix Q with columns $\vec{u}$, $\vec{v}$, $\vec{w}$. Similarly, we form a 3×3 matrix S with columns $\vec{a}$, $\vec{b}$, $\vec{a} \times \vec{b}$. Then we compute $R = SQ^\top$. That's it.

If you've studied linear algebra, then you should be able to check that this answer is correct pretty quickly. Try applying R to $\vec{u}$ and to $\vec{v}$, symbolically. You should get $\vec{a}$ and $\vec{b}$.

3 Describing Rotations In Terms Of Angle And Axis

Any rotation of three-dimensional space can be conceptualized as an axis of rotation and an amount of rotation about that axis. The axis can be described as a length-1, three-dimensional vector $\vec{u}$. The amount of rotation can be described as a number α, which is an angle measured in radians. The rotation rotates space about the axis $\vec{u}$, through the angle α, counter-clockwise in a right-handed sense. To understand what this means, hold your right hand in the air, with its fingers curled but its thumb pointing out. The thumb is the axis. If $\alpha > 0$, then the fingers point in the direction of rotation. If $\alpha < 0$, then the fingers point opposite to the direction of rotation. The amount of rotation is $|\alpha|$.

The rotation about $\vec{u}$ through angle α should be writable as a 3×3 matrix R. An explicit expression for this matrix was found by Euler and Rodrigues in the late 1700s and early 1800s. Here it is. First we place the three components of the vector $\vec{u}$ in a matrix U like this:

$$U = \begin{bmatrix} 0 & -u_2 & u_1 \\ u_2 & 0 & -u_0 \\ -u_1 & u_0 & 0 \end{bmatrix}.$$

Then we square that matrix U. The resulting matrix U^2 can be written in several ways:

$$U^2 = UU = \begin{bmatrix} -u_1{}^2 - u_2{}^2 & u_0u_1 & u_0u_2 \\ u_0u_1 & -u_0{}^2 - u_2{}^2 & u_1u_2 \\ u_0u_2 & u_1u_2 & -u_0{}^2 - u_1{}^2 \end{bmatrix} = \begin{bmatrix} u_0{}^2 - 1 & u_0u_1 & u_0u_2 \\ u_0u_1 & u_1{}^2 - 1 & u_1u_2 \\ u_0u_2 & u_1u_2 & u_2{}^2 - 1 \end{bmatrix}.$$

Then the desired rotation matrix is

$$R = I + (\sin\alpha)U + (1 - \cos\alpha)U^2.$$

In the equation above, the first term on the right is the 3×3 identity matrix. The second term is U with each of its entries scaled by $\sin\alpha$. The third term is U^2 with each of its entries scaled by $1 - \cos\alpha$. The three terms are added up, entry by entry, to produce R. In other words,

$$R_{ij} = I_{ij} + (\sin\alpha)U_{ij} + (1 - \cos\alpha)(U^2)_{ij}.$$

If you wish to understand better, how the Euler-Rodrigues formula works, then you might try computing the special case where $\alpha = 0$. You might also try the special case $u_0 = 0$, $u_1 = 0$, $u_2 = 1$. You might also try computing $R\vec{u}$ symbolically, under no special assumptions. Do the answers make sense?