This document teaches you the basics of vectors: scaling, addition, subtraction, and their geometric meaning. We work mainly in two dimensions, before shifting to arbitrary dimensions near the end.

1 What Is A Vector?

A *vector* is a quantity possessing both magnitude and direction. To emphasize that they are different from ordinary numbers, vectors are often written with arrows over them, as in $\vec{v}$.

For example, imagine that you live in a farm house on an open plain. Starting from your house, you walk 2 km northwest. Your displacement is a vector $\vec{d}$. Its magnitude is 2 km, and its direction is northwest. See Figure 1.

Let's emphasize that the vector does not describe your starting point or your ending point, but rather the difference between the two. If you start from a nearby pond and walk 2 km northwest from there, then your displacement is exactly the same vector $\vec{d}$ as in the previous example.

2 Coordinates

It is extremely common to pick some "standard" directions and express all vectors in terms of those directions. Consider again our hypothetical farm house. Let's standardize on the directions east and north. If we walk 3 km east, then our displacement is the vector

$$\vec{d} = \begin{bmatrix} 3 \\ 0 \end{bmatrix},$$

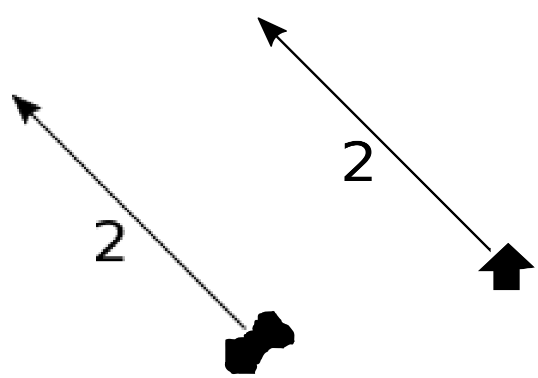

Figure 1: A displacement vector $\vec{d}$ representing a walk of 2 km to the northwest, starting from either a farm house or a nearby pond. (Can you perceive that one of the arrows is rasterized and the other is not? Did I do that for educational purposes? No, I'm just terrible at this drawing program.)

meaning "3 km east and 0 km north". If we walk 5 km south, then our displacement is

$$\vec{d} = \begin{bmatrix} 0 \\ -5 \end{bmatrix},$$

meaning "0 km east and -5 km north". And what about our original example, of walking 2 km northwest? Using the Pythagorean theorem, we figure out that this displacement is equivalent to $1/\sqrt{2}$ km west and $1/\sqrt{2}$ km north. So it is the vector

$$\vec{d} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}.$$

In all cases, the first number in $\vec{d}$ is the east-coordinate and the second number is the north-coordinate. We denote these numbers d_0 and d_1 respectively, and

$$\vec{d} = \begin{bmatrix} d_0 \\ d_1 \end{bmatrix}.$$

At this point you might be thinking, "Isn't it easier to think about 2 km northwest than about $-1/\sqrt{2}$ km east and $1/\sqrt{2}$ km north?" Well, yes. But in the long term vectors are easier to manipulate in terms of coordinates.

3 Scaling

Continuing our farm example, suppose that you take a walk with displacement

$$\vec{d} = \begin{bmatrix} 3 \\ 4 \end{bmatrix},$$

which is of length 5 in a northeast-ish direction. And the next day you walk in the same direction but twice as far. Then the second day's displacement is

$$\vec{d} = \begin{bmatrix} 6 \\ 8 \end{bmatrix},$$

because you're going twice as far east and twice as far north, right?

In general, if you have a number c and a vector $\vec{v}$, then the *scalar multiple* $c\vec{v}$ is

$$c\vec{v} = c \begin{bmatrix} v_0 \\ v_1 \end{bmatrix} = \begin{bmatrix} cv_0 \\ cv_1 \end{bmatrix}.$$

This $c\vec{v}$ has the same direction as $\vec{v}$, but its magnitude is $|c|$ times as big, and it might be pointing in the opposite direction. Here are some cases stated precisely.

- If $c > 1$, then $c\vec{v}$ is a stretched version of $\vec{v}$.

- If $c = 1$, then $c\vec{v} = \vec{v}$. Scaling by 1 has no effect.
- If $0 < c < 1$, then $c\vec{v}$ is a shrunken version of $\vec{v}$.
- If $c = 0$, then $c\vec{v} = \vec{0}$, which is the vector of length 0.
- If $-1 < c < 0$, then $c\vec{v}$ points in the opposite direction from $\vec{v}$, and is shrunken.
- If $c = -1$, then $c\vec{v} = -\vec{v}$, meaning the vector of the same length as $\vec{v}$ but pointing in the opposite direction.
- If $c < -1$, then $c\vec{v}$ points in the opposite direction from $\vec{v}$, and is stretched.

4 Addition

Continuing our farm example, suppose that you take a walk with displacement

$$\vec{d} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

on one day, camp there overnight, and then walk

$$\vec{e} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

the next day. What is your total displacement from your house? You have moved $2 + 1 = 3$ km in the east direction and $1 + 0 = 1$ km in the north direction, for a total displacement of

$$\vec{d} + \vec{e} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

The left side of Figure 2 is a diagram of this two-day walk. The tail of $\vec{e}$ is placed at the head of $\vec{d}$. The sum $\vec{d} + \vec{e}$ then runs from the tail of $\vec{d}$ to the head of $\vec{e}$. The right side of the figure shows what happens if you do the walk in the opposite order: first $\vec{e}$ then $\vec{d}$. You get the same total displacement vector. The order of vector addition doesn't matter.

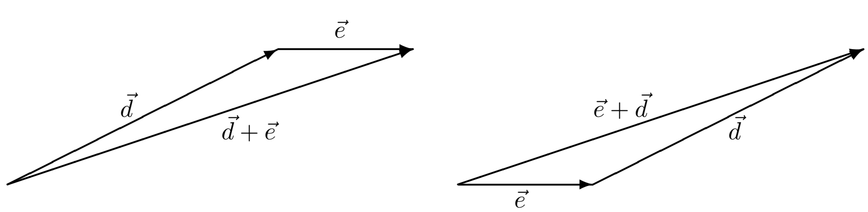

Figure 2: The left side shows the total displacement arising from displacement $\vec{d}$ followed by displacement $\vec{e}$. The right side shows $\vec{e}$ followed by $\vec{d}$. The overall effect is the same.

In general, if you have two vectors $\vec{v}$ and $\vec{w}$, then the *sum* $\vec{v} + \vec{w}$ is

$$\vec{v} + \vec{w} = \begin{bmatrix} v_0 \\ v_1 \end{bmatrix} + \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} = \begin{bmatrix} v_0 + w_0 \\ v_1 + w_1 \end{bmatrix}.$$

Let's confirm, algebraically, that adding in the opposite order gives the same result:

$$\vec{w} + \vec{v} = \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} + \begin{bmatrix} v_0 \\ v_1 \end{bmatrix} = \begin{bmatrix} w_0 + v_0 \\ w_1 + v_1 \end{bmatrix}.$$

Yep, that's the same result, because $v_0 + w_0 = w_0 + v_0$ and $v_1 + w_1 = w_1 + v_1$.

Addition is where is starts to become clear, why the coordinate description of vectors is best. Think about how you would perform addition, if you instead recorded vectors as lengths and directions. To implement addition, you'd have to do some trigonometry, to figure out the length and direction of the sum $\vec{v} + \vec{w}$. Even once you figured out how to solve the problem in general, your resulting program would be drastically more complicated than it needed to be, and slow and numerically imprecise too.

5 Algebraic Rules

Vectors obey most of the rules of algebra that you learned years ago in school. Here is a logically precise summary. There exists a vector $\vec{0}$ such that, for all numbers b and c, and for all vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$:

- $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$ (additive associativity),
- $\vec{v} + \vec{0} = \vec{v} = \vec{0} + \vec{v}$ (additive identity),
- there exists $-\vec{v}$ such that $\vec{v} + -\vec{v} = \vec{0} = -\vec{v} + \vec{v}$ (additive inverses),
- $\vec{v} + \vec{w} = \vec{w} + \vec{v}$ (additive commutativity),
- $b(c\vec{v}) = (bc)\vec{v}$ (scalar associativity),
- $1\vec{v} = \vec{v}$ (scalar identity),
- $(b + c)\vec{v} = b\vec{v} + c\vec{v}$ (distributivity over scalar addition),
- $c(\vec{v} + \vec{w}) = c\vec{v} + c\vec{w}$ (distributivity over vector addition).

Many other rules follow logically from the ones listed above, including $0\vec{v} = \vec{0}$ and $-1\vec{v} = -\vec{v}$.

6 Subtraction

What does it mean to subtract two vectors, as in $\vec{v} - \vec{w}$? Based on the rules of algebra, a reasonable guess is that

$$\vec{v} - \vec{w} = \vec{v} + (-\vec{w}).$$

Another reasonable guess is that $\vec{v} - \vec{w}$ is the vector that, when you add it to $\vec{w}$, gives you $\vec{v}$:

$$(\vec{v} - \vec{w}) + \vec{w} = \vec{v}.$$

Based on coordinates, another reasonable guess is that $\vec{v} - \vec{w}$ is the vector that you get by subtracting coordinates:

$$\vec{v} - \vec{w} = \begin{bmatrix} v_0 \\ v_1 \end{bmatrix} - \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} = \begin{bmatrix} v_0 - w_0 \\ v_1 - w_1 \end{bmatrix}.$$

All of these guesses give the same answer, and that's what vector subtraction means.

7 Points As Vectors

Very often we speak of points using vector notation, as in "the point $\vec{p}$". In what way can a vector $\vec{p}$ be considered a point? Well, a vector can be thought of as an arrow from one point (the vector's tail) to another point (the head). So is one of those the point that we want? No! Remember from Figure 1 that the vector can be moved around, changing its tail and head, without changing as a vector.

When we speak of vectors as points, it's because we have chosen a center of the universe, relative to which points are being measured. We place the vector's tail at the center of the universe. The point in question is then the vector's head.

In practice, this is all pretty easy. The center of the universe is typically the origin of a coordinate system, and the vector

$$\vec{p} = \begin{bmatrix} p_0 \\ p_1 \end{bmatrix}$$

points from the origin to the point (p_0, p_1).

Here's a question to test your understanding: If I have vectors $\vec{p}$ and $\vec{q}$, when what is the vector that points from point $\vec{p}$ to point $\vec{q}$? Draw a picture to figure it out.

8 In Arbitrary Dimensions

Thus far, all of our examples have occurred in a two-dimensional plane (or, metaphorically, a plain). However, vectors can be defined in any dimension. For example, if a bird flies 7 km east and 5 km south, and meanwhile gains 1 km of altitude, then its three-dimensional displacement is

$$\vec{d} = \begin{bmatrix} d_0 \\ d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} 7 \\ -5 \\ 1 \end{bmatrix}$$

in the east-north-up coordinate system.

Here are some examples of how vectors of various dimensions are used in this course.

- A two-dimensional vector can describe a point on the computer screen. For example, the three vertices of a triangle are three two-dimensional vectors.
- There are three components to an RGB color. We can think of them as one three-dimensional vector. This vector represents a point in a virtual space of colors.
- When we design virtual 3D worlds, we need to specify the locations of objects in those worlds. Those locations are three-dimensional point vectors.
- When we perform viewing transformations on those three-dimensional point vectors, they become four-dimensional, for technical reasons to be explained later.
- A piece of software called the vertex shader outputs vectors of dimension `varyDim`. What is this dimension? We don't even know, because the user of our graphics engine writes the vertex shader and chooses `varyDim`. In one of my demos, `varyDim` ends up being 11.

So you might find yourself working with vectors of high dimension. That might be scary, because you can't visualize high dimensions (and neither can I). But the good news is that you can trust the algebra. All of the arithmetic operations behave as you would expect, following the rules laid out earlier. In particular,

$$\begin{aligned} \vec{v} + \vec{w} &= \begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_{n-1} \end{bmatrix} + \begin{bmatrix} w_0 \\ w_1 \\ \vdots \\ w_{n-1} \end{bmatrix} = \begin{bmatrix} v_0 + w_0 \\ v_1 + w_1 \\ \vdots \\ v_{n-1} + w_{n-1} \end{bmatrix}, \\ c\vec{v} &= c \begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_{n-1} \end{bmatrix} = \begin{bmatrix} cv_0 \\ cv_1 \\ \vdots \\ cv_{n-1} \end{bmatrix}. \end{aligned}$$

Finally, you might be thinking, "Wouldn't indices $1, 2, \ldots, n$ be simpler to use than indices $0, 1, \ldots, n-1$?" Well, math courses do typically index from 1. But we're going to be doing a lot of computations with vectors stored as coordinate arrays. And arrays in C are indexed from 0. So my tutorials index from 0, to remove one source of bugs.