Poisson processes are used to model a wide variety of phenomena in nature and society. Poisson processes are closely related to two probability distributions: the Poisson distribution (which is discrete) and the exponential distribution (which is continuous).

1 Poisson distribution

Definition: A discrete random variable X is *Poisson-distributed* with parameter $\lambda > 0$, written $X \sim \text{Pois}(\lambda)$, if

$$P(X = k) = \frac{e^{-\lambda}\lambda^k}{k!}$$

for $k = 0, 1, 2, \ldots$ (and $P(X = k) = 0$ for other k).

Exercise 1. *Verify that the PMF sums to* 1*.*

Exercise 2. *Compute the expectation* $E(X)$*.*

Exercise 3. *Compute the variance* $V(X)$*. (Hint: Start with a power series that showed up in the solution to the previous exercise. Differentiate it. It will help you compute* $E(X^2)$*.)*

2 Poisson processes

Definition: A *Poisson process* with rate $\lambda > 0$ is a set of occurrences (or successes, or arrivals) in continuous time, such that the number of arrivals in an interval of length t is $\text{Pois}(\lambda t)$, and the numbers of arrivals in disjoint intervals are independent of each other.

In a Poisson process, the number of occurrences in a fixed time period is a Poisson-distributed X. Usually, we have some data about the average number of occurrences per time period, which helps us determine the value of the parameter λ. The numerical value of λ depends on the unit of time chosen, so we should always be explicit and careful about the time unit.

Many practical situations can be modeled as Poisson processes. For an example, let's talk about lightning strikes in Minnesota. Assume that the number of strikes each year doesn't depend on the number of strikes during the previous year, whether we measure years from January 1 to December 31 or from July 1 to June 30, etc. Then the number of strikes, that will occur next year, is Poisson-distributed. Meteorological records let us estimate λ.

Exercise 4. *Over the past* 43 *years, the USA has experienced* 33 *"major" (meaning magnitude* 7 *or greater, I think) earthquakes. Suppose that these earthquakes constitute a Poisson process. Then a Poisson-distributed* X *is lurking in the problem. What is the meaning of* X*? What is the value of* λ*? What are the meaning and the value of* $P(X \geq 2)$*?*

Exercise 5. *The 1990s book and film A Civil Action tell a true story about the town of Woburn, Massachusetts. Residents of the town sue a local company, because they suspect that the company's pollution is causing elevated cancer rates in the town.*

At the time, the USA had about 280,000,000 *people and* 30,800 *leukemia cases annually. So there were* $30{,}800/280{,}000{,}000 \approx 0.00011$ *cases per person. Now consider a town of population* 35,000*. Assuming that leukemia rates are the same in this town as in the USA, how many cases does one expect in the town this year? What is the probability that the town has* 8 *or more cases this year?*

3 Relationship to the binomial distribution

A binomially distributed random variable Y counts the number of successes in n independent Bernoulli trials, each of success probability p. Recall that $E(Y) = np$. Now one might ask, "How many successes will occur in infinitely many trials?" The answer should be something like $\infty p = \infty$. So it's not a good question.

However, there is a way to tweak the question so that it gives a sensible, finite answer. As we let the number n of trials go to infinity, we balance it by letting the success probability p go to zero. More precisely, fix a number λ and set $p = \lambda/n$ in the binomial:

$$\begin{aligned} P(Y = k) &= \binom{n}{k} p^k (1-p)^{n-k} \\ &= \frac{n!}{k!(n-k)!} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k} \\ &= \frac{\lambda^k}{k!} \frac{n(n-1)\cdots(n-k+1)}{n^k} \left(1 - \frac{\lambda}{n}\right)^n \left(1 - \frac{\lambda}{n}\right)^{-k}. \end{aligned}$$

Exercise 6. *As $n \to \infty$, what happens to each of the four factors in the expression above?*

The preceding exercise should show that the binomial distribution limits to the Poisson distribution. Practically speaking, when n is large, p is small, and $\lambda = pn$ is medium-sized, the Poisson distribution is a good approximation to the binomial distribution. In other words, the Poisson distribution is useful for modeling rare events.

Exercise 7. *In physics, a particle of one type can spontaneously decay to a particle of another type. The decay is equally probable at all instants of time and for all particles of the first type. Commonly the decays are modeled as a Poisson process.*

We begin with n particles of the first type. After an hour, we observe that ϵn of them have decayed, where ϵ is a certain number between 0 *and* 1*. Let X be the number of decays observed in the next hour.*

1. *If $\epsilon = 0.5$, then does it seem reasonable to model X as Poisson?*

2. If $\epsilon = 0.000001$, then does it seem reasonable to model X as Poisson?

4 Relationship to the exponential distribution

In a Poisson process, we expect λ arrivals per unit of time. Divide a unit time interval into $n \gg \lambda$ sub-intervals. Then $\lambda/n \ll 1$ arrivals are expected per sub-interval. We can regard each sub-interval as a Bernoulli trial with $p = \lambda/n$ (so that the expectations agree). These Bernoulli trials are independent, by the definition of Poisson process. Then the number of failures before the first success is $X \sim \text{Geom}(p)$. Let $Y = X \cdot \frac{1}{n}$, so that Y is the time elapsed before the first success. Then

$$\begin{aligned}
P(Y \geq y) &= P(X/n \geq y) \\
&= P(X \geq ny) \\
&\approx (1-p)^{ny} \qquad \text{(equal if } ny \text{ is an integer)} \\
&= \left(\left(1 - \frac{\lambda}{n}\right)^n\right)^y.
\end{aligned}$$

Exercise 8. *In the limit $n \to \infty$, what is $P(Y \geq y)$? So what are the CDF and PDF of Y?*

5 Exponential distribution

Definition: A continuous random variable Y is *exponentially distributed* with parameter $\lambda > 0$, written $Y \sim \text{Expo}(\lambda)$, if $f_Y(y) = \lambda e^{-\lambda y}$ for $y > 0$.

The preceding section shows that an exponentially distributed Y measures a Poisson process's "inter-arrival times" — that is, how much time elapses between the start of the process and the first occurrence, or between one occurrence and the next. In a sense, the exponential distribution is a continuous analogue of the geometric distribution.

Exercise 9. *Verify that the PDF integrates to* 1.

Exercise 10. *Compute the expectation $E(Y)$.*

Exercise 11. *Compute the variance $V(Y)$.*

Exercise 12. *You run the emergency room in a hospital in a big city. You get* 10 *patients per hour, on average. Patient arrivals can be modeled as a Poisson process.*

A. Find λ, and pose questions answered by $X \sim \text{Pois}(\lambda)$ and $Y \sim \text{Expo}(\lambda)$.

B. What's the probability that no patients arrive in the next hour? Compute it in two ways: using X and using Y.

Exercise 13. *A hard disk drive is a mechanical device that rotates thousands of times per second, often for years on end. Consequently it lasts only a few years before breaking. A data center has* 100,000 *drives. Every* 15 *minutes, on average, one of the drives fails and needs to be replaced. Repeat the preceding exercise in this new context. In fact, do it twice, using two different units of time.*

6 Miscellany

Exercise 14. *Let* $Y \sim \text{Expo}(\lambda)$*. Let* s *and* t *be numbers such that* $t > s > 0$*. Compute* $P(Y \geq t | Y \geq s)$*. Explain why this result makes intuitive sense.*

Exercise 15. *Suppose that* $X_1 \sim \text{Pois}(\lambda_1)$ *and* $X_2 \sim \text{Pois}(\lambda_2)$ *are independent. Find the PMF of* $X_1 + X_2$*. Do you recognize the distribution of* $X_1 + X_2$*?*

Exercise 16. *Suppose that* $Y_1, Y_2, \ldots, Y_n$ *are independent, and* $Y_i \sim \text{Expo}(\lambda_i)$*. Let* $L = \min(Y_1, Y_2, \ldots, Y_n)$*. Then it turns out that* $L \sim \text{Expo}(\lambda_1 + \lambda_2 + \cdots + \lambda_n)$*. (I am not asking you to prove this fact.) Explain why this result makes intuitive sense.*

In a Poisson process, the inter-arrival times are exponentially distributed, but the arrival times themselves follow the gamma distribution. We don't study the gamma distribution in this course, but read Section 7.2, if you're interested.