This document reviews some facts and techniques from calculus, that we use in Math 240. I omit some technical details, because they don't arise in Math 240. There are some exercises at the end. Let me know if you want to discuss any of this material.

1 Derivatives

The derivative of a function f is another function, which is denoted f' or $\frac{d}{dx}f$. The derivative tells us, at any point of f's graph, what the slope of the tangent line is. The act of computing derivatives is called differentiation. Here are the four most important general rules of differentiation:

1. Sum rule: $(f+g)' = f' + g'$. For example,
$$\frac{d}{dx}\left(x^2 + x^3\right) = \frac{d}{dx}\left(x^2\right) + \frac{d}{dx}\left(x^3\right) = 2x + 3x^2.$$
2. Scaling rule: $(cf)' = cf'$. For example,
$$\frac{d}{dx}(7\sin x) = 7\frac{d}{dx}(\sin x) = 7\cos x.$$
3. Product rule: $(fg)' = f'g + fg'$. For example,
$$\frac{d}{dx}(x\cos x) = \frac{d}{dx}(x)\cos x + x\frac{d}{dx}(\cos x) = \cos x - x\sin x.$$
4. Chain rule: $(f \circ g)' = (f' \circ g)g'$. For example,
$$\frac{d}{dx}\left(\sin\left(x^3\right)\right) = \cos\left(x^3\right)3x^2.$$

Here, we have also used a couple of specific examples, such as $\frac{d}{dx}\sin x = \cos x$, $\frac{d}{dx}\cos x = -\sin x$, and the power rule, which says that $\frac{d}{dx}\left(x^k\right) = kx^{k-1}$ for any constant $k \neq 0$.

We might need to optimize functions occasionally. If f is continuous on a closed, finite interval $[a, b]$, then it attains a maximum value and a minimum value. Further, those optima can occur only at these four kinds of places: at a, at b, where f' is undefined, and where f' is zero. So we find those places, evaluate f at them, and compare those f-values to determine the optima.

2 Anti-derivatives

The anti-derivative of a function f is denoted $\int f \; dx$. It is a function, or rather a class of functions — namely, all of the functions whose derivatives are f. It turns out that they all differ by additive constants. I mean, if F and G are two anti-derivatives of f, then there is some constant C such that $F = G + C$.

The four basic differentiation rules have corresponding anti-differentiation rules. Two of their names are weird, for historical reasons:

1. Sum rule: $\int f + g\ dx = \int f\ dx + \int g\ dx$. For example,
$$\int x^2 + x^3\ dx \quad = \quad \int x^2\ dx + \int x^3\ dx \quad = \quad \frac{1}{3}x^3 + \frac{1}{4}x^4 + C.$$
2. Scaling rule: $\int cf\ dx = c \int f\ dx$. For example,
$$\int 7 \sin x\ dx \quad = \quad 7 \int \sin x\ dx \quad = \quad -7 \cos x + C.$$
3. Parts: $fg + C = \int f'g + fg'\ dx$, or equivalently $\int fg'\ dx = fg - \int f'g\ dx$. For example,
$$\int -x \sin x\ dx \quad = \quad x \cos x - \int \cos x\ dx \quad = \quad x \cos x - \sin x + C.$$
4. Substitution: $f \circ g + C = \int (f' \circ g) g'\ dx$. For example,
$$\int \cos\left(x^3\right) 3x^2\ dx \quad = \quad \sin\left(x^3\right) + C.$$

3 Integrals

The integral of a function f on an interval $[a, b]$ is denoted $\int_a^b f\ dx$. It is a number — namely, the area trapped between the graph of f and the x-axis, over the interval $[a, b]$. Conceptually, integrals are computed by adding up areas of tall, skinny boxes. Let me emphasize: Although the integral $\int_a^b f\ dx$ looks similar to the anti-derivative $\int f\ dx$ notationally, they are different kinds of objects, and there is no obvious relationship between them.

Shockingly, the fundamental theorem of calculus says that there is a relationship between integrals and anti-derivatives: If F is any anti-derivative of f, then

$$\int_a^b f\ dx \quad = \quad F(b) - F(a). \tag{1}$$

Practically, this is how we compute integrals in a course like Math 240.

If $f(x, y)$ is a function of two variables, then its graph is a surface floating above the x-y-plane in three-dimensional space. Let R be a region in the x-y-plane. The double integral $\iint_R f(x, y)\ dA$ is the volume trapped between the graph of f and the x-y-plane, over the region R. Conceptually, double integrals are computed by adding up volumes of tall, skinny boxes. Practically, we compute double integrals as iterated integrals.

The simplest case is when R is the rectangle formed by an interval $[a,b]$ in the x-axis "times" an interval $[c,d]$ in the y-axis. Then

$$\iint_R f(x,y)\;dA \quad = \quad \int_a^b \int_c^d f(x,y)\;dy\;dx.$$

The iterated integral on the right is computed from the inside out. That is, first we compute the inside integral, $\int_c^d f(x,y)\;dy$, regarding x as a constant. In doing so, we obtain an expression that (usually) has some xs in it. Call it $F(x)$. Then we compute $\int_a^b F(x)\;dx$. Here's a worked example:

$$\begin{aligned}
\int_0^1 \int_0^1 x + 3y\;dy\;dx \quad &= \quad \int_0^1 \left[xy + \frac{3}{2}y^2\right]_{y=0}^1 \;dx \\
&= \quad \int_0^1 \left[\left(x + \frac{3}{2}\right) - (0+0)\right]\;dx \\
&= \quad \int_0^1 x + \frac{3}{2}\;dx \\
&= \quad \left[\frac{1}{2}x^2 + \frac{3}{2}x\right]_0^1 \\
&= \quad \left(\frac{1}{2} + \frac{3}{2}\right) - (0+0) \\
&= \quad 2.
\end{aligned}$$

In this example, $x + 3/2$ is what I called $F(x)$ above.

For a more difficult example, let R be the triangle whose vertices are $(0,0)$, $(1,0)$, and $(0,1)$. We have to figure out which points (x,y) lie inside R, and encode those points into the bounds of the iterated integral. The relevant x-values run from 0 rightward to 1. For any one of those x-values, the relevant y-values run from 0 upward to the diagonal line $y = 1 - x$. Therefore

$$\iint_R f(x,y)\;dA \quad = \quad \int_0^1 \int_0^{1-x} f(x,y)\;dy\;dx.$$

Again, we compute from the inside out. The inside integral $\int_0^{1-x} f(x,y)\;dy$ produces some expression $F(x)$, whose xs arise either from those already in f or from plugging in the bound $1 - x$. Once we have $F(x)$, we compute $\int_0^1 F(x)\;dx$.

When the region R is rotationally symmetric about the origin, a double integral may be more easily computed in polar coordinates. For any point (x,y) in the plane, the polar coordinate $r = \sqrt{x^2 + y^2}$ is the distance to the origin, and the polar coordinate θ indicates the direction from the origin to (x,y). Specifically, θ is an angle measured counter-clockwise from the positive x-axis to the line segment between the origin and (x,y). To be integrated in polar coordinates, the integrand $f(x,y)$ must be re-expressed in terms of r and θ, using the substitutions $x = r\cos\theta$ and $y = r\sin\theta$. Moreover, the integrand must be multiplied by r to account for the non-linear area distortion between (x, y) coordinates and (r, θ) coordinates. For example, suppose that we want to integrate $f(x, y) = x^2 + y^2 + y$ over the entire x-y-plane. So the region R is where $r \geq 0$ and $0 \leq \theta \leq 2\pi$, and $f(r\cos\theta, r\sin\theta) = r^2 + r\sin\theta$. The integral is

$$\int_0^{2\pi} \int_0^{\infty} (r^2 + r\sin\theta) r \, dr \, d\theta.$$

That iterated integral is computed from the inside out, as in the examples above. (The answer is infinite. Try to come up with an example that is not infinite.)

4 Series

Let r be a constant. The geometric series with common ratio r is

$$1 + r + r^2 + r^3 + \cdots \quad = \quad \sum_{k=0}^{\infty} r^k.$$

When $|r| \geq 1$, the series does not have a value. When $|r| < 1$, the value is

$$\sum_{k=0}^{\infty} r^k \quad = \quad \frac{1}{1-r}. \tag{2}$$

Another series is the exponential function

$$\exp(x) \quad = \quad e^x \quad = \quad \sum_{k=0}^{\infty} \frac{x^k}{k!} \quad = \quad 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots.$$

That's an example of a power series. The general form of a power series is

$$f(x) \quad = \quad \sum_{k=0}^{\infty} a_k x^k \quad = \quad a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots,$$

where the coefficients a_k are constants. The derivative can be computed term-by-term:

$$f'(x) \quad = \quad a_1 + 2a_2 x + 3a_3 x^2 + \cdots \quad = \quad \sum_{k=1}^{\infty} k a_k x^{k-1}.$$

Notice that the new series starts at $k = 1$ rather than $k = 0$. If we differentiate again, we get

$$f''(x) \quad = \quad 2a_2 + 6a_3 x + \cdots \quad = \quad \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2}.$$

If we prefer our series to start at $k = 0$, then we can always re-index. For example, let $\ell = k - 2$. Then

$$f''(x) \quad = \quad \sum_{\ell=0}^{\infty} (\ell+2)(\ell+1) a_{\ell+2} x^{\ell} \quad = \quad 2a_2 + 6a_3 x + \cdots.$$

We can even rename ℓ to k (because the old k is no longer in the problem, and the names of these counter variables are arbitrary). So

$$f''(x) \quad = \quad \sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k \quad = \quad 2a_2 + 6a_3 x + \cdots.$$

5 Miscellany

There is a second, equivalent way to define the exponential function:

$$\exp(x) \quad = \quad e^x \quad = \quad \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n .$$

This definition turns out to be useful in some problems of computing limits.

There is a second, equivalent formulation of the fundamental theorem of calculus. For any constant a,

$$\frac{d}{dx} \int_a^x f(t)\, dt = f(x). \tag{3}$$

What if the upper bound is not just x, but some function $g(x)$? Then you get a fancier version:

$$\frac{d}{dx} \int_a^{g(x)} f(t)\, dt = f(g(x)) \cdot g'(x). \tag{4}$$

6 Exercises

Exercise 1 *Combining rules of differentiation stated above, show that the quotient rule is true:*

$$\left(\frac{f}{g}\right)' \quad = \quad \frac{f'g - fg'}{g^2}.$$

Exercise 2 *Let $f(x) = (1-x)^{25}x$. Find the value of x where f is maximized, on the interval* $[0, 1]$.

Exercise 3 *Assuming that the first form of the fundamental theorem (Eqn. 1) is true, show that the second form (Eqn. 3) is true. Conversely, assuming the second form, show that the first form is true. (This work does not show that they're both true. It merely shows that they're both true or both false. They're both true, but I'm not asking you to prove that.) Then, assuming the second form, show that the modified second form (Eqn. 4) is true.*

Exercise 4 *Let $f(x, y) = xy$. Compute the double integral of f over the rectangle bounded by $x = 0$, $x = 3$, $y = 1$, and $y = 2$. Then compute the double integral of f over the triangle with vertices* $(0, 1)$, $(3, 1)$, *and* $(3, 2)$.

Exercise 5 *Show that Eqn. 2 gives the value of the geometric series. (Hint: Start by computing* $(1-r)(1 + r + r^2 + \cdots + r^{n-1} + r^n)$ *for a positive integer n.)*

Exercise 6 *Show that* $\frac{d}{dx}\exp(x) = \exp(x)$.

Exercise 7 *Show that*

$$\sum_{k=0}^{\infty} k^2 q^k = \frac{q(q+1)}{(1-q)^3}$$

for any $0 < q < 1$. (Hint: Start with a geometric series. Differentiate, multiply through by q, differentiate again, and multiply through by q again.)