A.A. [Draw a picture of the support of the joint distribution.] The support of X is $[0,1]$. On that support,

$$f_X(x) = \int_0^x 8xy\, dy = \left[4xy^2\right]_0^x = 4x^3.$$

[Check: That PDF integrates to 1 on $[0,1]$.]

A.B. The conditional expectation is

$$\begin{aligned} E(Y|X=x) &= \int_{-\infty}^{\infty} y f_{Y|X}(y|x)\, dy \\ &= \int_{-\infty}^{\infty} y \frac{f_{X,Y}(x,y)}{f_X(x)}\, dy \\ &= \int_0^x y \frac{8xy}{4x^3}\, dy \\ &= \int_0^x 2y^2x^{-2}\, dy \\ &= \left[\frac{2}{3}y^3x^{-2}\right]_0^x \\ &= \frac{2}{3}x. \end{aligned}$$

A.C. Because $E(Y|X=x) = \frac{2}{3}x$, we immediately conclude that $E(Y|X) = \frac{2}{3}X$.

A.D. The conditional variance is

$$\begin{aligned} V(Y|X=x) &= \int_{-\infty}^{\infty} (y - E(Y|X=x))^2\, f_{Y|X}(y|x)\, dy \\ &= \int_0^x \left(y - \frac{2}{3}x\right)^2 \frac{8xy}{4x^3}\, dy \\ &= \int_0^x \left(y^2 - \frac{4}{3}xy + \frac{4}{9}x^2\right) 2yx^{-2}\, dy \\ &= \int_0^x 2y^3x^{-2} - \frac{8}{3}x^{-1}y^2 + \frac{8}{9}y\, dy \\ &= \left[\frac{1}{2}y^4x^{-2} - \frac{8}{9}x^{-1}y^3 + \frac{4}{9}y^2\right]_0^x \\ &= \frac{1}{2}x^2 - \frac{8}{9}x^2 + \frac{4}{9}x^2 \\ &= \frac{1}{18}x^2. \end{aligned}$$

B.A. The MGF of X is

$$m_X(t) = E(e^{tX}) = \sum_{k=0}^{1} e^{tk} P(X=k) = e^0(1-p) + e^t p = 1 - p + pe^t.$$

B.B. We know that $Y = X_1 + \cdots X_n$, where all $X_i \sim \text{Bern}(p)$ are independent. Therefore

$$m_Y(t) = m_{X_1+\cdots+X_n}(t) = m_{X_1}(t) \cdot \cdots \cdot m_{X_n}(t) = m_{X_i}(t)^n = (1 - p + pe^t)^n.$$

B.C. The expectation of Y is the first moment of Y, which is $m'_Y(0)$. So we compute

$$m'_Y(t) = n(1 - p + pe^t)^{n-1} \cdot pe^t \quad \Rightarrow \quad m'_Y(0) = n(1 - p + p)^{n-1} \cdot p = np.$$

[Check: Yes, np matches what we know to be the expectation of a binomial random variable.]

C.A. By the definition of Poisson process, the X_i are independent and identically distributed, with $X_i \sim \text{Expo}(\lambda)$. So $E(X_i) = 1/\lambda$, $V(X_i) = 1/\lambda^2$,

$$E(S_n) = E(X_1 + \cdots + X_n) = E(X_1) + \cdots + E(X_n) = n/\lambda,$$

and

$$V(S_n) = V(X_1 + \cdots + X_n) = V(X_1) + \cdots + V(X_n) = n/\lambda^2.$$

When n is large, the central limit theorem says that S_n is approximately normal. So, approximately, $S_n \sim \text{Norm}(n/\lambda, n/\lambda^2)$.

C.B. The standard deviation of S_n is $\sqrt{n}/\lambda$, and we know that 95% of the mass of the normal distribution is within two standard deviations of the mean. So, when n is large: "The probability that S_n is between $n/\lambda - 2\sqrt{n}/\lambda$ and $n/\lambda + 2\sqrt{n}/\lambda$ is approximately 0.95."

D.A. [We did this entire problem, nearly verbatim, in class recently.] The conventional wisdom is that we want to be generating lots of value and making lots of money, which means great $E(S_n)$. However, for stability and predictability, we want small $V(S_n)$.

D.B. First,

$$E(S_n) = E(X_1 + \cdots + X_n) = E(X_1) + \cdots + E(X_n) = n\mu,$$

by linearity of expectation, which doesn't require independence. Second,

$$V(S_n) = V(X_1 + \cdots + X_n) = V(X_1) + \cdots + V(X_n) = n\sigma^2,$$

by independence.

D.C. We know that

$$\text{Cov}(X_i, X_j) \leq \text{SD}(X_i) \cdot \text{SD}(X_j) = \sigma^2.$$

Then

$$\begin{aligned}
V(S_n) &= V(X_1 + \cdots + X_n) \\
&= \sum_{i=1}^{n} V(X_i) + 2\sum_{i<j} \mathrm{Cov}(X_i, X_j) \\
&\leq n\sigma^2 + 2 \cdot \binom{n}{2} \cdot \sigma^2 \\
&= \sigma^2(n + n(n-1)) \\
&= n^2\sigma^2.
\end{aligned}$$

D.D. Under independence, we have variance $n\sigma^2$. Under dependence, the variance could be as large as $n^2\sigma^2$. So it seems that policy makers should encourage independence.

More precisely, policy makers should discourage positive correlation. It might actually be nice to have negative correlation, if one could engineer it. However, negative correlation can at best reduce the possible variance from $n\sigma^2$ to 0.

E. Let D be the event that you have the disease, and let T be the event that you test positive. We are told that $P(D) = 0.01$, $P(T^c|D) = 0.08$, and $P(T|D^c) = 0.02$. Then the probability that you have the disease, given that you've tested negative, is

$$\begin{aligned}
P(D|T^c) &= \frac{P(T^c|D)P(D)}{P(T^c)} \\
&= \frac{P(T^c|D)P(D)}{P(T^c|D)P(D) + P(T^c|D^c)P(D^c)} \\
&= \frac{0.08 \cdot 0.01}{0.08 \cdot 0.01 + (1 - 0.02) \cdot 0.99}.
\end{aligned}$$

[By the way, the answer is approximately 0.0008238929.]