A. The easiest computation is $E(X)$, because it requires only an anti-differentiation by substitution:

$$E(X) = \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx = \left[-\frac{1}{\sqrt{2\pi}} e^{-x^2/2}\right]_{-\infty}^{\infty} = -0 - -0 = 0.$$

B. Intuitively, they are not independent, because knowing the value of Y changes our predictions for the value of $X+Y$. For example, if we know that $Y = 8$, then we suspect that $X + Y$ is near 8. For a clearer, more convincing argument, notice that

$$\mathrm{Cov}(X+Y, Y) = \mathrm{Cov}(X, Y) + \mathrm{Cov}(Y, Y) = 0 + V(Y) = 1.$$

Because $X + Y$ and Y have non-zero covariance, they cannot be independent.

C.A. First I compute the probability of no 100-year floods in the next 30 years. If our time unit is the year, then $\lambda = 0.01$, and $Y \sim \mathrm{Expo}(\lambda)$ measures the waiting time until the first 100-year flood. So I want

$$P(Y \geq 30) = e^{-\lambda \cdot 30} = e^{-0.3}.$$

Alternatively, if our time unit is the thirty-year, then $\lambda = 0.3$, and $X \sim \mathrm{Pois}(\lambda)$ counts how many 100-year floods occur in the next 30 years. So I want

$$P(X = 0) = e^{-\lambda}\lambda^0/0! = e^{-0.3}.$$

Either way, the answer to the question is $1 - e^{-0.3}$. [By the way, this is approximately 26%.]

C.B. How many 100-year floods will happen in the next century? The answer is $X \sim \mathrm{Pois}(1)$, where the unit of time is the century.

D. The support of Z is $[2, \infty)$. We start doing convolution:

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z - x)\,dx.$$

We need to encode the support of X and Y into the bounds. First, $f_X(x) > 0$ only when $x \geq 1$. Second, $f_Y(z - x) > 0$ only when $z - x \geq 1$, or in other words when $x \leq z - 1$. Therefore

$$f_Z(z) = \int_1^{z-1} x^{-2}(z-x)^{-2}\,dx = \int_1^{z-1} \frac{1}{x^2(z-x)^2}\,dx.$$

We've never done an integral like that in Math 240, so we stop there. [Students who reached this point earned full credit. There is an algorithm called partial fractions for computing the required anti-derivative. It's tedious and error-prone, so I had Mathematica do it. Anyway, the answer is

$$f_Z(z) = \left[\frac{z(z-2x) + 2x(x-z)\log(x/(x-z))}{x(x-z)z^3}\right]_1^{z-1} = \frac{2z(z-2) + 4\log(1-z)}{z^3(z-1)}.$$

In Mathematica, I then checked that the integral of this expression, from 2 to ∞, is indeed 1. I think that it requires parts and then partial fractions.]

E.A. [Draw a picture!] No, X and Y are not independent. For example, knowing that $X = 2$ tells us that $Y = 1$. Also, while it is true that the joint PDF splits into a function of x and a function of y on its support, the support doesn't have the necessary rectangle shape, to make the PDF split everywhere.

E.B. The picture helps us determine the bounds in

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\ dy = \int_{x/z}^{1} 2xy\ dy = \left[xy^2\right]_{x/2}^{1} = x - x^3/4$$

on support $[0, 2]$.

E.C. Again, the picture helps us determine the support of $f_{Y|X}$ to be $[x/2, 1]$. On that support,

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{2xy}{x - x^3/4} = \frac{2y}{1 - x^2/4}.$$