A1. By a routine calculation, curl $\vec{F} = \langle -y, -1, 1 - 3x^2 \cos y \rangle$.

A2. By a routine calculation, div $\vec{F} = z + 4z^3 + 6x \sin y$.

A3. Because div(curl $\vec{G}$) $= 0$ for any smooth vector field $\vec{G}$, we can conclude without any computation that div(curl $\vec{F}$) $= 0$.

A4. No, $\vec{F}$ is not conservative. If $\vec{F}$ were equal to grad f for some smooth f, then we would have curl $\vec{F} = \text{curl}(\text{grad } f) = \vec{0}$. But curl $\vec{F} \neq \vec{0}$, so $\vec{F}$ cannot be conservative.

B1. The function to be optimized is $f(x, y, z) = (a - hy)x + (b - kz)y + cz$, the constraint is $g(x, y, z) = x + y + z = d$, and their gradients are

$$\nabla f = \begin{bmatrix} a - hy \\ -hx + b - kz \\ -ky + c \end{bmatrix}, \quad \nabla g = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Therefore we obtain four equations

$$\begin{aligned} a - hy &= \lambda, \\ -hx + b - kz &= \lambda, \\ -ky + c &= \lambda, \\ x + y + z &= d \end{aligned}$$

in the four unknowns x, y, z, λ.

B2. The first and third equations imply that $y = \frac{a-c}{h-k}$ and $\lambda = \frac{ch-ak}{h-k}$. Then the second and fourth equations imply that

$$\begin{aligned} x &= \frac{bh - bk - ch - ck + 2ak - dhk + dk^2}{(h-k)^2}, \\ z &= \frac{-bh + bk - ah - ak + 2ch + dh^2 - dhk}{(h-k)^2}. \end{aligned}$$

C. The Earth's mass is the integral of the Earth's density over the region E of space occupied by the Earth:

$$\begin{aligned}
\iiint_E Ad^2/R^2 + B\,dV &= \int_0^{2\pi}\int_0^{\pi}\int_0^R (A\rho^2/R^2 + B)\rho^2 \sin\phi\,d\rho\,d\phi\,d\theta \\
&= \int_0^{2\pi} d\theta \cdot \int_0^{\pi} \sin\phi\,d\phi \cdot \int_0^R (A\rho^2/R^2 + B)\rho^2 d\rho \\
&= 4\pi \int_0^R \frac{A}{R^2}\rho^4 + B\rho^2\,d\rho \\
&= 4\pi \left[\frac{A}{5R^2}\rho^5 + \frac{B}{3}\rho^3\right]_0^R \\
&= 4\pi \left(\frac{A}{5} + \frac{B}{3}\right) R^3.
\end{aligned}$$

[By the way, the Earth's true density function is more complicated than this. It decreases sharply at the transitions between the major layers of the Earth: the inner core, outer core, mantle, and crust. At each d, the density function above is within a factor of two of the true density. Therefore our calculated mass is within a factor of two of the true mass.]

D. Given $(a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n)$, let

$$f(x, y) = \sum_{i=1}^{n} (x - a_i)^2 + (y - b_i)^2.$$

Our goal is to minimize f over the entire x-y-plane. So we compute

$$\nabla f = \begin{bmatrix} \sum 2(x - a_i) \\ \sum 2(y - b_i) \end{bmatrix} = \begin{bmatrix} 2(nx - \sum a_i) \\ 2(ny - \sum b_i) \end{bmatrix}.$$

The gradient is never undefined. The gradient is zero exactly where $x = \frac{1}{n}\sum a_i$ and $y = \frac{1}{n}\sum b_i$.

Intuitively, this unique critical point must be the global minimum. There is no maximum, because f increases without bound as x and y grow large. Further, there can't fail to be a minimum, because $f \geq 0$ everywhere and f increases as x and y grow large. We can support that intuition with rigorous calculation. The second partial derivatives are $f_{xx} = f_{yy} = 2n$ and $f_{xy} = f_{yx} = 0$. Therefore $f_{xx} f_{yy} - f_{xy} f_{yx} = 4n^2 > 0$ and $f_{xx} > 0$. So the second derivative test confirms that the critical point is a minimum.

E. [By the way, this is Section 15.2, Exercise 29.] Based on a picture, which I'll not show here,

$$\begin{aligned}
\int_0^4 \int_{\sqrt{y}}^2 \sqrt{4x^2 + 5y}\, dx\, dy &= \int_0^2 \int_0^{x^2} \sqrt{4x^2 + 5y}\, dy\, dx \\
&= \int_0^2 \left[\frac{2}{15}(4x^2 + 5y)^{3/2} \right]_0^{x^2} dx \\
&= \int_0^2 \frac{2}{15}(4x^2 + 5x^2)^{3/2} - \frac{2}{15}(4x^2)^{3/2}\, dx \\
&= \int_0^2 \frac{38}{15} x^3\, dx \\
&= \left[\frac{19}{30} x^4 \right]_0^2 \\
&= 152/15.
\end{aligned}$$