A. Let $\vec{v} = (0, 2, 0) - (1, 0, 0) = \langle -1, 2, 0\rangle$ and $\vec{w} = (0, 0, 3) - (1, 0, 0) = \langle -1, 0, 3\rangle$. Then $\vec{n} = \vec{v} \times \vec{w} = \langle 6, 3, 2\rangle$ is perpendicular to the plane, with length 7. Thus $\vec{n}/|\vec{n}| = \langle 6/7, 3/7, 2/7\rangle$ is a unit vector perpendicular to the plane. [The negation of that answer is an equally good answer.]

B. By stretching our usual circle parametrization, we can parametrize the ellipse as $\vec{c}(t) = (2 \cos t, 3 \sin t)$. Notice that $\vec{c}\,'(t) = \langle -2 \sin t, 3 \cos t\rangle$ is tangent to the ellipse and hence $\vec{n} = \langle -3 \cos t, -2 \sin t\rangle$ is normal to the ellipse. Notice also that $\vec{n}$ points "into" the curve of the ellipse, and hence is a positive multiple of the normal vector $\vec{N}$. Because $|\vec{n}| = \sqrt{9 \cos^2 t + 4 \sin^2 t} = \sqrt{4 + 5 \cos^2 t}$, we conclude that

$$\vec{N} = \frac{\langle -3 \cos t, -2 \sin t\rangle}{\sqrt{4 + 5 \cos^2 t}}.$$

C. [This is similar to a homework problem. Specifically, this problem relates to Day 22 Problem B exactly as Day 24 Problem B relates to Day 22 Problem A.] Recall from homework the product rule for divergence:

$$\mathrm{div}(f\vec{F}) = \nabla f \cdot \vec{F} + f\mathrm{div}\vec{F}.$$

Therefore, for a region W of 3D space,

$$\iiint_W \mathrm{div}(f\vec{F})\, dV = \iiint_W \nabla f \cdot \vec{F}\, dV + \iiint_W f\mathrm{div}\vec{F}\, dV.$$

By the divergence theorem, the term on the left equals $\iint_{\partial W} (f\vec{F}) \cdot d\vec{S}$. Rearranging the terms a bit, we have an integration by parts formula

$$\iiint_W f\mathrm{div}\vec{F}\, dV = \iint_{\partial W} (f\vec{F}) \cdot d\vec{S} - \iiint_W \nabla f \cdot \vec{F}\, dV.$$

D1. Well,

$$\begin{aligned}
\text{curl }(\text{curl }\vec{F}) &= \begin{bmatrix} \partial_x \\ \partial_y \\ \partial_z \end{bmatrix} \times \left(\begin{bmatrix} \partial_x \\ \partial_y \\ \partial_z \end{bmatrix} \times \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} \right) \\
&= \begin{bmatrix} \partial_x \\ \partial_y \\ \partial_z \end{bmatrix} \times \begin{bmatrix} \partial_y F_3 - \partial_z F_2 \\ \partial_z F_1 - \partial_x F_3 \\ \partial_x F_2 - \partial_y F_1 \end{bmatrix} \\
&= \begin{bmatrix} \partial_{yx} F_2 - \partial_{yy} F_1 - \partial_{zz} F_1 + \partial_{zx} F_3 \\ \partial_{zy} F_3 - \partial_{zz} F_2 - \partial_{xx} F_2 + \partial_{xy} F_1 \\ \partial_{xz} F_1 - \partial_{xx} F_3 - \partial_{yy} F_3 + \partial_{yz} F_2 \end{bmatrix} \\
&= \begin{bmatrix} \partial_{xx} F_1 + \partial_{xy} F_2 + \partial_{xz} F_3 - \partial_{xx} F_1 - \partial_{yy} F_1 - \partial_{zz} F_1 \\ \partial_{yx} F_1 + \partial_{yy} F_2 + \partial_{yz} F_3 - \partial_{zz} F_2 - \partial_{yy} F_2 - \partial_{xx} F_2 \\ \partial_{zx} F_1 + \partial_{zy} F_2 + \partial_{zz} F_3 - \partial_{xx} F_3 - \partial_{yy} F_3 - \partial_{zz} F_3 \end{bmatrix} \\
&= \begin{bmatrix} \partial_x(\text{div}\vec{F}) - \Delta F_1 \\ \partial_y(\text{div}\vec{F}) - \Delta F_2 \\ \partial_z(\text{div}\vec{F}) - \Delta F_3 \end{bmatrix} \\
&= \nabla(\text{div}\vec{F}) - \Delta \vec{F}.
\end{aligned}$$

D2. Taking the curl of Maxwell's third equation and using problem D1, we have

$$\nabla(\text{div}\vec{E}) - \Delta\vec{E} = \text{curl }\left(-\frac{\partial \vec{B}}{\partial t}\right).$$

On the left side, the first term vanishes because of Maxwell's first equation. On the right side, $\frac{\partial}{\partial t}$ commutes with curl, because they involve different derivatives. [Compute this out if you like.] Therefore

$$-\Delta\vec{E} = -\frac{\partial}{\partial t}\text{curl }\vec{B}.$$

Plugging Maxwell's fourth equation into the right side produces the wave equation.

E. Let $f(\alpha, \beta) = \sin\alpha + \sin\beta + \sin(\pi - \alpha - \beta)$. The first and second partial derivatives are

$$\begin{aligned}
\frac{\partial f}{\partial \alpha} &= \cos\alpha - \cos(\pi - \alpha - \beta), \\
\frac{\partial f}{\partial \beta} &= \cos\beta - \cos(\pi - \alpha - \beta), \\
\frac{\partial}{\partial \alpha}\frac{\partial f}{\partial \alpha} &= -\sin\alpha - \sin(\pi - \alpha - \beta), \\
\frac{\partial}{\partial \beta}\frac{\partial f}{\partial \beta} &= -\sin\beta - \sin(\pi - \alpha - \beta), \\
\frac{\partial}{\partial \beta}\frac{\partial f}{\partial \alpha} &= -\sin(\pi - \alpha - \beta).
\end{aligned}$$

The critical points occur where $\frac{\partial f}{\partial \alpha} = \frac{\partial f}{\partial \beta} = 0$, so where $\cos\alpha = \cos(\pi - \alpha - \beta) = \cos\beta$. Because α, β, and $\pi - \alpha - \beta$ are all non-negative, the unique critical point is

$$\alpha = \beta = \pi/3 = \pi - \alpha - \beta.$$

Now we perform the second derivative test. At the critical point, $\frac{\partial}{\partial \alpha}\frac{\partial f}{\partial \alpha} = \frac{\partial}{\partial \beta}\frac{\partial f}{\partial \beta} = -\sqrt{3}$ and $\frac{\partial}{\partial \beta}\frac{\partial f}{\partial \alpha} = -\sqrt{3}/2$. Because $\frac{\partial}{\partial \alpha}\frac{\partial f}{\partial \alpha}\frac{\partial}{\partial \beta}\frac{\partial f}{\partial \beta} - \left(\frac{\partial}{\partial \beta}\frac{\partial f}{\partial \alpha}\right)^2 = 3 - 3/4 > 0$, the critical point is a local maximum or minimum. Because $\frac{\partial}{\partial \alpha}\frac{\partial f}{\partial \alpha} < 0$, it must be a local maximum. The value of f at this point is $3\sqrt{3}/2$. If we insist that $\alpha, \beta, \gamma > 0$, then we are finished, because the domain of optimization has no boundary. If we allow one of the angles to degenerate to 0, then the other two angles go to $\pi/2$, and f has the value 2, which is less than its value at the critical point. Therefore, even if we consider degenerate triangles, f is maximized at $\alpha = \beta = \gamma = \pi/3$.

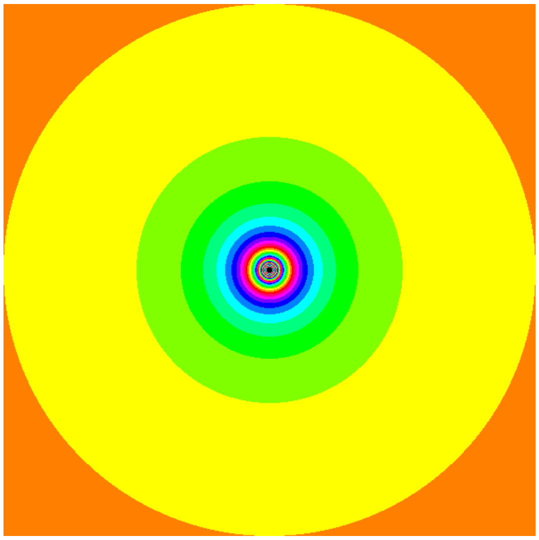
F1. Well,

$$z^4 = ((x + iy)^2)^2 = ((x^2 - y^2) + i(2xy))^2 = ((x^2 - y^2)^2 - 4x^2y^2) + i(4(x^2 - y^2)xy).$$

Therefore the vector field is

$$\langle (x^2 - y^2)^2 - 4x^2y^2 + c_1, 4(x^2 - y^2)xy + c_2 \rangle.$$

F2. The black part of the fractal, representing those values of $\vec{c}$ for which $(0, 0)$ never escapes, consists of just the origin $(0, 0)$. Every other point in the plane is non-black. These points are colored according to their distance from the origin, with the colors changing more quickly as we approach the origin. Here is a plot of the fractal in $[-2, 2] \times [-2, 2]$.



G. [I'll omit the sketch.] The integral to compute is

$$\begin{aligned}\int_0^1 \int_{x^2}^x \int_0^x x + 2y \, dz \, dy \, dx &= \int_0^1 \int_{x^2}^x x^2 + 2xy \, dy \, dx \\ &= \int_0^1 \left[x^2 y + xy^2\right]_{y=x^2}^{y=x} \, dx \\ &= \int_0^1 x^3 + x^3 - x^4 - x^5 \, dx \\ &= \left[\frac{1}{2}x^4 - \frac{1}{5}x^5 - \frac{1}{6}x^6\right]_0^1 \\ &= \frac{1}{2} - \frac{1}{5} - \frac{1}{6} \\ &= \frac{2}{15}.\end{aligned}$$

H. Well,

$$\begin{aligned}-\frac{1}{\rho}\nabla p + \nabla \cdot T &= \begin{bmatrix} -\frac{1}{\rho}\partial_x p + \partial_x T_{11} + \partial_y T_{12} + \partial_z T_{13} \\ -\frac{1}{\rho}\partial_y p + \partial_x T_{21} + \partial_y T_{22} + \partial_z T_{23} \\ -\frac{1}{\rho}\partial_z p + \partial_x T_{31} + \partial_y T_{32} + \partial_z T_{33} \end{bmatrix} \\ &= \begin{bmatrix} \partial_x (T_{11} - \frac{1}{\rho}p) + \partial_y T_{12} + \partial_z T_{13} \\ \partial_x T_{21} + \partial_y (T_{22} - \frac{1}{\rho}p) + \partial_z T_{23} \\ \partial_x T_{31} + \partial_y T_{32} + \partial_z (T_{33} - \frac{1}{\rho}p) \end{bmatrix} \\ &= \nabla \cdot U,\end{aligned}$$

where

$$U = \begin{bmatrix} T_{11} - \frac{1}{\rho}p & T_{12} & T_{13} \\ T_{21} & T_{22} - \frac{1}{\rho}p & T_{23} \\ T_{31} & T_{32} & T_{33} - \frac{1}{\rho}p \end{bmatrix}.$$

I. Let S be the portion of the ellipsoid $(x/4)^2 + (y/3)^2 + (z/2)^2 = 1$ where $x, y, z \leq 0$. Orient S so that it has upward-pointing normals. Compute the flux of $\vec{F} = \langle 0, 0, z \rangle$ across S.

We parametrize the ellipsoid by

$$\vec{G}(\phi, \theta) = (4 \sin \phi \cos \theta, 3 \sin \phi \sin \theta, 2 \cos \phi).$$

Then

$$\begin{aligned}
\vec{G}_\phi &= \langle 4 \cos \phi \cos \theta, 3 \cos \phi \sin \theta, -2 \sin \phi \rangle, \\
\vec{G}_\theta &= \langle -4 \sin \phi \sin \theta, 3 \sin \phi \cos \theta, 0 \rangle, \\
\vec{n} &= \vec{G}_\theta \times \vec{G}_\phi \\
&= \langle -6 \sin^2 \phi \cos \theta, 8 \sin^2 \phi \sin \theta, -12 \sin \phi \cos \phi \rangle.
\end{aligned}$$

Let's check that we have oriented $\vec{n}$ correctly. For example, the point $(-4, 0, 0)$ is on S. At that point, $\phi = \pi/2$ and $\theta = \pi$, so $\vec{n} = \langle 6, 0, 0 \rangle$. This normal vector points "into" the ellipsoid, and hence $\vec{n}$ is upward-pointing on S. The flux is

$$\begin{aligned}
\iint_S \vec{F} \cdot d\vec{S} &= \int_\pi^{3\pi/2} \int_{\pi/2}^\pi \vec{F}(\vec{G}(\phi, \theta)) \cdot \vec{n}(\phi, \theta) \, d\phi \, d\theta \\
&= \int_\pi^{3\pi/2} \int_{\pi/2}^\pi 2 \cos \phi \cdot -12 \sin \phi \cos \phi \, d\phi \, d\theta \\
&= -24 \frac{\pi}{2} \int_{\pi/2}^\pi \cos^2 \phi \sin \phi \, d\phi \\
&= -12\pi \left[-\frac{1}{3} \cos^3 \phi \right]_{\pi/2}^\pi \\
&= 4\pi \left(\cos^3 \pi - \cos^3 \frac{\pi}{2} \right) \\
&= -4\pi.
\end{aligned}$$

Let's check that the sign is correct. The normal $\vec{n}$ points up, while $\vec{F}$ points down. So we expect the flux to be negative.