A.A. $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ i\frac{1}{\sqrt{2}} \end{bmatrix} \otimes \begin{bmatrix} \frac{\sqrt{3}}{2} - \frac{1}{2}i \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \cdot (\frac{\sqrt{3}}{2} - \frac{1}{2}i) \\ \frac{1}{\sqrt{2}} \cdot 0 \\ i\frac{1}{\sqrt{2}} \cdot (\frac{\sqrt{3}}{2} - \frac{1}{2}i) \\ i\frac{1}{\sqrt{2}} \cdot 0 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}}i \\ 0 \\ \frac{1}{2\sqrt{2}} + \frac{\sqrt{3}}{2\sqrt{2}}i \\ 0 \end{bmatrix}.$

A.B. $X \otimes H = \begin{bmatrix} 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \end{bmatrix}.$

B.A. α is a label or name for a quantum state or any complex vector. In this example, we are defining a linear transformation according to its effect on the standard basis, so α is the name of a standard basis vector $|\alpha\rangle$ in $\mathbb{C}^2$. So $\alpha = 0$ or $\alpha = 1$. More generally, if $|\alpha\rangle$ were a standard basis vector in $\mathbb{C}^{2^n}$, then α would be an n-bit bit string.

B.B. $|\alpha\rangle$ is one of the standard basis vectors of $\mathbb{C}^2$. So $|\alpha\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ or $|\alpha\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

B.C. The first f is a two-qbit quantum gate. That is, it is a unitary linear transformation of $\mathbb{C}^4$ or equivalently a 4×4 unitary matrix.

B.D. The second f is a classical one-bit function $f : \{0, 1\} \to \{0, 1\}$. There are four possibilities for what f is.

C.A. When $|\alpha\rangle$ is measured, the state changes to $|0\rangle$ with probability $\frac{1}{2}$ and to $|1\rangle$ with probability $\frac{1}{2}$. Exactly the same answer holds for $|\beta\rangle$. They are both uniform superpositions of the classical one-qbit states.

C.B. Here is a quantum algorithm that behaves differently on $|\alpha\rangle$ than on $|\beta\rangle$: Multiply the state by H and then measure. If the state is $|\alpha\rangle$, then the measurement certainly produces $|0\rangle$. If the state is $|\beta\rangle$, then the measurement certainly produces $|1\rangle$.

D.A. [I'll omit the drawing from these solutions. It should have, from left to right, two wires crossing, then a CNOT gate, then two wires crossing.]

D.B. The matrices for SWAP and CNOT are, respectively,

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Therefore the matrix for SWAP · CNOT · SWAP is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

which we have seen as the upside-down CNOT.

E.A. TRUE. [The exponential map wraps the imaginary axis around the unit circle infinitely many times.]

E.B. TRUE. [If U is unitary, then $U^{-1} = U^*$.]

E.C. TRUE. [If the two-qbit state $|\chi\rangle$ is classical, then three of its entries are 0, so it satisfies the unentanglement condition $\chi_{00}\chi_{11} = \chi_{01}\chi_{10}$.]

E.D. FALSE. [Two of the classical one-bit gates are non-invertible and hence cannot be implemented as one-qbit gates.]

E.E. FALSE. [The two-bit AND (or NAND, or OR, or NOR) gate cannot be implemented as a two-qbit gate.]

E.F. TRUE. [Any two-qbit state is a linear combination of classical two-qbit states.]

E.G. FALSE. [Partial measurement makes one of the qbits classical, but not necessarily the other.]

E.H. TRUE. [And one of the qbits is also classical.]

F.A. Deutsch's problem is: Given a two-qbit gate that implements one of the four classical one-bit functions f (in the usual $|\alpha\rangle |\beta\rangle \mapsto |\alpha\rangle |\beta \oplus f(\alpha)\rangle$ way), determine whether the hidden function f is constant or non-constant.

F.B. Deutsch's algorithm is: Compute $(H \otimes H) \cdot f \cdot (H \otimes H) \cdot (X \otimes X) |0\rangle |0\rangle$. Then measure the first qbit. If it is $|0\rangle$, then f is non-constant. If it is $|1\rangle$, then f is constant.