A. [This is an easier version of page 1063 #54. I will omit the drawing, but you should not.] The region of integration is like a curvy pyramid with its vertex at the origin and its base along the $y = 2$ plane. The five vertices are at $(0, 0, 0)$, $(8, 2, 0)$, $(0, 2, 0)$, $(8, 2, 4)$, and $(0, 2, 4)$. The desired integral is

$$\int_0^4 \int_{\sqrt{z}}^2 \int_0^{y^3} f(x, y, z)\, dx\, dy\, dz.$$

B. We want to compute

$$\begin{aligned}\iiint_{\mathbb{R}^3} e^{-(x^2+y^2+z^2)}\, dV &= \int_0^{2\pi} \int_0^{\pi} \int_0^{\infty} e^{-\rho^2} \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta \\ &= \left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\pi} \sin\phi\, d\phi\right)\left(\int_0^{\infty} e^{-\rho^2}\rho^2 d\rho\right) \\ &= 2\pi \cdot 2 \cdot \lim_{b\to\infty} \int_0^b e^{-\rho^2}\rho^2 d\rho.\end{aligned}$$

Integration by parts tells us that

$$\int e^{-\rho^2}\rho^2\, d\rho = -\frac{1}{2}\rho e^{-\rho^2} + \frac{1}{2}\int e^{-\rho^2}\, d\rho.$$

Therefore the triple integral is

$$\begin{aligned}2\pi \lim_{b\to\infty}\left(\left[-\rho e^{-\rho^2}\right]_0^b + \int_0^b e^{-\rho^2}\, d\rho\right) &= 2\pi \lim_{b\to\infty} -be^{-b^2} + 2\pi\int_0^{\infty} e^{-\rho^2}\, d\rho \\ &= 2\pi \lim_{b\to\infty} \frac{-b}{e^{b^2}} + \pi\int_{-\infty}^{\infty} e^{-\rho^2}\, d\rho \\ &= 2\pi \lim_{b\to\infty} \frac{-1}{2be^{b^2}} + \pi K \\ &= \pi K.\end{aligned}$$

(We have used the fact that $e^{-\rho^2}$ is an even function of ρ, the definition of K, and L'Hopital's Rule.)

[Bonus problem: Working in Cartesian coordinates, argue that the value of the triple integral is K^3. Then deduce the value of K.]

C. Because $\vec{r}(t) = \left\langle \frac{1}{2}t^2, \frac{\sqrt{2}}{3}t^3, \frac{1}{4}t^4 \right\rangle$, we have $\vec{r}'(t) = \langle t, \sqrt{2}t^2, t^3\rangle$ and

$$|\vec{r}'(t)| = \sqrt{t^2 + 2t^4 + t^6} = \sqrt{t^2(1+t^2)^2} = t(1+t^2) = t + t^3,$$

because $t \geq 0$. Then the arc length is

$$\int_0^1 |\vec{r}'(t)|\, dt = \int_0^1 t + t^3\, dt = \left[\frac{1}{2}t^2 + \frac{1}{4}t^4\right]_0^1 = \frac{3}{4}.$$

D. [This is a slightly easier version of page 1149 #9.] The vector field is not conservative (why?), so we cannot use the fundamental theorem of calculus for line integrals. Also, the curve is not closed, so we cannot use Green's theorem. So we proceed by brute force. The work is

$$\begin{aligned}
\int_0^1 \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\, dt &= \int_0^1 \langle xe^z, xz, x+y \rangle \cdot \langle 1, 3t^2, -2t \rangle\, dt \\
&= \int_0^1 \langle te^{-t^2}, -t^3, t+t^3 \rangle \cdot \langle 1, 3t^2, -2t \rangle\, dt \\
&= \int_0^1 te^{-t^2} - 3t^5 - 2t^2 - 2t^4\, dt \\
&= \left[-\frac{1}{2}e^{-t^2} - \frac{1}{2}t^6 - \frac{2}{3}t^3 - \frac{2}{5}t^5 \right]_0^1 \\
&= -\frac{1}{2}e^{-1} - \frac{1}{2} - \frac{2}{3} - \frac{2}{5} - -\frac{1}{2}e^0 \\
&= -\frac{1}{2}e^{-1} - \frac{2}{3} - \frac{2}{5} \\
&= -\frac{1}{2}e^{-1} - \frac{16}{15}.
\end{aligned}$$

E. [This is a slightly harder version of page 1149 #16. I urge you to draw C and the region that it encloses.] Let D be the triangular region enclosed by C. By Green's theorem,

$$\begin{aligned}
\int_C \sqrt{1+x^3}\, dx + 2xy\, dy &= \int_C \left\langle \sqrt{1+x^3}, 2xy \right\rangle \cdot d\vec{s} \\
&= \iint_D \frac{\partial}{\partial x} 2xy - \frac{\partial}{\partial y} \sqrt{1+x^3}\, dA \\
&= \int_0^1 \int_0^{3x} 2y\, dy\, dx \\
&= \int_0^1 \left[y^2 \right]_0^{3x}\, dx \\
&= \int_0^1 9x^2\, dx \\
&= \left[3x^3 \right]_0^1 \\
&= 3.
\end{aligned}$$