A. This vector field $\vec{F}$ has curl zero but is not conservative — that is, $\vec{F}$ does not arise as the gradient of any potential function. Or, if you don't like the concept of curl, we can instead say that the cross-partials condition $\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$ is met, but $\vec{F}$ is not conservative.

This state of affairs is possible only because the domain of $\vec{F}$ is not simply connected. If $\vec{F}$ were defined at the origin and had curl zero there, then $\vec{F}$ would have to be conservative.

B. It is helpful to consider (and even diagram) how the variables depend on each other. The function f depends on t, x, and y. But x and y depend on t and ϵ. Thus the total derivative of f with respect to t is

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}(a_1'(t) + \epsilon b_1'(t)) + \frac{\partial f}{\partial y}(a_2'(t) + \epsilon b_2'(t)),$$

and the total derivative of f with respect to ϵ is

$$\frac{\partial f}{\partial x}\frac{\partial x}{\partial \epsilon} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \epsilon} = \frac{\partial f}{\partial x} b_1(t) + \frac{\partial f}{\partial y} b_2(t).$$

The critical points of f, regarded as a function of t and ϵ, occur where both of these quantities are zero or undefined. Because all functions are smooth, the quantities are never undefined. Therefore the critical points occur where

$$\begin{aligned} \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}(a_1'(t) + \epsilon b_1'(t)) + \frac{\partial f}{\partial y}(a_2'(t) + \epsilon b_2'(t)) &= 0, \\ \frac{\partial f}{\partial x} b_1(t) + \frac{\partial f}{\partial y} b_2(t) &= 0. \end{aligned}$$

C. [This problem is 17.1 Exercise 35.] By the definition of flux, the definition of $\vec{n}$, a little algebra, and Green's theorem, the flux equals

$$\begin{aligned} \int_a^b \vec{F}(\vec{c}(t)) \cdot \vec{n}(t)\, dt &= \int_a^b \langle F_1, F_2 \rangle \cdot \langle y', -x' \rangle\, dt \\ &= \int_a^b \langle -F_2, F_1 \rangle \cdot \langle x', y' \rangle\, dt \\ &= \int_{\partial D} \langle -F_2, F_1 \rangle \cdot d\vec{s} \\ &= \iint_D \frac{\partial F_1}{\partial x} - -\frac{\partial F_2}{\partial y}\, dA \\ &= \iint_D \text{div } \vec{F}\, dA. \end{aligned}$$

D. Let C be the curve parametrized by $\vec{c}$. Then by the definition of work and the fundamental theorem for line integrals, the work performed by $\vec{F}$ is

$$\int_C \vec{F} \cdot d\vec{s} = -\int_C \nabla V \cdot d\vec{s} = -(V(3,5,9) - V(1,1,1)) = kq_1q_2 \left(\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{115}} \right).$$

[Let's check that the sign makes sense. If q_1 and q_2 have the same sign, then the force should be repulsive, so $\vec{F}$ wants them to move away from each other. That is, $\vec{F}$ helps the second particle move from $\vec{P}$ to $\vec{Q}$, so the work performed by $\vec{F}$ should be positive. Yep.]

E. [This problem is 15.3 Exercise 31, which was assigned as homework. I'll omit the sketch.] The volume is

$$\begin{aligned}
\iiint_W 1\,dV &= \int_{-1}^{1}\int_{x^2}^{1}\int_{0}^{1-y^2} 1\,dz\,dy\,dx \\
&= \int_{-1}^{1}\int_{x^2}^{1} 1-y^2\,dy\,dx \\
&= \int_{-1}^{1} \left[y-y^3/3\right]_{x^2}^{1}\,dx \\
&= \int_{-1}^{1} \left(1-\frac{1}{3}\right)-\left(x^2-\frac{1}{3}x^6\right)\,dx \\
&= \left[\frac{2}{3}x-\frac{1}{3}x^3+\frac{1}{21}x^7\right]_{-1}^{1} \\
&= \frac{16}{21}.
\end{aligned}$$

F. [This problem is 14.8 Exercise 27, which was assigned as homework.] Let $f(x,y,z) = \sqrt{x^2+y^2+z^2}$ and $g(x,y,z) = ax+by+cz$. We wish to minimize f subject to $g = d$. Notice that, because f is nonnegative, minimizing f is equivalent to minimizing $f^2 = x^2+y^2+z^2$, which is simpler than f. Following the Lagrange multipliers approach, we compute $\nabla f^2 = \langle 2x, 2y, 2z\rangle$ and $\nabla g = \langle a, b, c\rangle$. We wish to solve

$$\begin{aligned}
2x &= \lambda a, \\
2y &= \lambda b, \\
2z &= \lambda c, \\
ax+by+cz &= d.
\end{aligned}$$

The last equation can be rewritten, with the help of the first three equations, as

$$\frac{\lambda}{2}(a^2+b^2+c^2) = d.$$

Solving for λ in that equation, and plugging the resulting expression for λ into the first three equations, yields

$$(x,y,z) = \frac{d}{a^2+b^2+c^2}(a,b,c).$$

This is the only solution to the four equations. Intuitively, there is a minimum distance from the origin to the plane, so this solution must produce that minimum. The distance from the origin to this point is

$$f\left(\frac{d}{a^2+b^2+c^2}(a,b,c)\right) = \left|\frac{d}{a^2+b^2+c^2}\right|\sqrt{a^2+b^2+c^2} = \frac{|d|}{\sqrt{a^2+b^2+c^2}}.$$

G1. Integrating in polar coordinates, we have

$$\begin{aligned}\iint_D \log(x^2+y^2)\,dA &= \int_0^{2\pi}\int_{R_1}^{R_2}\log(r^2)r\,dr\,d\theta\\ &= \frac{1}{2}\int_0^{2\pi}d\theta\int_{R_1}^{R_2}2r\log(r^2)\,dr\\ &= \pi\left[r^2\log(r^2)-r^2\right]_{R_1}^{R_2}\\ &= \pi R_2^2(\log(R_2^2)-1)-\pi R_1^2(\log(R_1^2)-1).\end{aligned}$$

G2. This problem is just like the previous problem, but with $R_2 = R$ and $R_1 \to 0$. So we compute

$$\lim_{R_1\to 0}\pi R^2(\log(R^2)-1)-\pi R_1^2(\log(R_1^2)-1) = \pi R^2(\log(R^2)-1)-\pi\lim_{R_1\to 0}R_1^2\log(R_1^2).$$

Now we focus on that limit term, which is of the form

$$\begin{aligned}\lim_{x\to 0}x^2\log(x^2) &= \lim_{x\to 0}\frac{\log(x^2)}{x^{-2}}\\ &= \lim_{x\to 0}\frac{x^{-2}2x}{-2x^{-3}}\\ &= \lim_{x\to 0}-x^2\\ &= 0\end{aligned}$$

by L'Hopital's rule. Therefore the value of the integral in question is $\pi R^2(\log(R^2)-1)$.