A.

$\vec{v} + \vec{w} = \langle 3, 4, 2, 3\rangle$.

$4\vec{v} = \langle 8, 4, -4, 4\rangle$.

$\vec{v} \cdot \vec{w} = 2 + 3 - 3 + 2 = 4$.

$\text{proj}_{\vec{w}}\vec{v} = \frac{\vec{v}\cdot\vec{w}}{\vec{w}\cdot\vec{w}}\vec{w} = \frac{4}{23}\langle 1, 3, 3, 2\rangle$.

B. If $f = -GM(x^2 + y^2 + z^2)^{-1/2}$, then

$$f_x = -GM(-1/2)(x^2 + y^2 + z^2)^{-3/2} 2x = GM(x^2 + y^2 + z^2)^{-3/2} x.$$

The other partials f_y and f_z are similar, with the consequence that

$$-\nabla f = -GM(x^2 + y^2 + z^2)^{-3/2}\vec{x} = \frac{GM}{|\vec{x}|^2} \cdot \frac{-\vec{x}}{|\vec{x}|}.$$

Here, $-\vec{x}/|\vec{x}|$ is the unit vector pointing from $\vec{x}$ toward the origin. So the acceleration has magnitude $GM/|\vec{x}|^2$ in that direction.

C1. Let $\vec{r}(t) = (t, f(t), 0)$. Then $\vec{r}'(t) = (1, f'(t), 0)$, $\vec{r}''(t) = (0, f''(t), 0)$, and

$$\kappa = \frac{|\vec{r}' \times \vec{r}''|}{|\vec{r}'|^3} = \frac{|(0, 0, f''(t))|}{|(1, f'(t), 0)|^3} = \frac{|f''(t)|}{(1 + (f'(t))^2)^{3/2}}.$$

C2. Notice that the curve lies entirely in the x-y-plane. Hence its unit tangent vector $\vec{T}$ is always in that plane, the derivative of $\vec{T}$ is always in that plane, and so is the unit normal $\vec{N} = \vec{T}'/|\vec{T}'|$. (This result matches our intuition, that $\vec{N}$ always points "into the turn" of the curve. For that to be true of a plane curve, $\vec{N}$ must lie in the plane.) Thus the unit binormal $\vec{B} = \vec{T} \times \vec{N}$ is $\langle 0, 0, \pm 1\rangle$ wherever it is defined, and $d\vec{B}/ds = \vec{0}$ wherever it exists. The torsion τ, being defined by $d\vec{B}/ds = -\tau\vec{N}$, must be 0 everywhere it exists.

D. We first express the locations of the two cities in spherical coordinates on a sphere of radius 1. Johannesburg is at $\phi = 116°$, $\theta = 28°$, while Phnom Penh is at $\phi = 78°$, $\theta = 105°$. Converting to Cartesian coordinates, we have Johannesburg at $\vec{j} = (\sin 116° \cos 28°, \sin 116° \sin 28°, \cos 116°)$ and Phnom Penh at $\vec{p} = (\sin 78° \cos 105°, \sin 78° \sin 105°, \cos 78°)$. Then

$$\vec{j} \cdot \vec{p} = |\vec{j}||\vec{p}| \cos\alpha = \cos\alpha,$$

where α is the central angle between the two vectors. Finally, the distance along the sphere in km is 6371α, which equals

$$6371 \arccos(\sin 116° \cos 28° \sin 78° \cos 105° + \sin 116° \sin 28° \sin 78° \sin 105° + \cos 116° \cos 78°).$$

E. In two dimensions we have polar coordinates

$$\begin{aligned} x &= \cos\theta, \\ y &= \sin\theta \end{aligned}$$

on the unit circle, and in three dimensions we have spherical coordinates

$$\begin{aligned} x &= \sin\phi\cos\theta, \\ y &= \sin\phi\sin\theta, \\ z &= \cos\phi \end{aligned}$$

on the unit sphere. Continuing this pattern, we try

$$\begin{aligned} x &= \sin\psi\sin\phi\cos\theta, \\ y &= \sin\psi\sin\phi\sin\theta, \\ z &= \sin\psi\cos\phi, \\ w &= \cos\psi \end{aligned}$$

on the unit hypersphere in four dimensions. One can check that $x^2+y^2+z^2+w^2 = 1$, as desired. One can also check that all regions of the sphere are covered, although that is harder to do. By the way, here is another answer:

$$\begin{aligned} x &= \sin\phi\cos\theta, \\ y &= \sin\phi\sin\theta, \\ z &= \cos\phi\cos\psi, \\ w &= \cos\phi\sin\psi. \end{aligned}$$

F1. Let $f(x,y) = \frac{x^3y}{x^6+y^2}$. Along the line $y = mx$,

$$f = \frac{mx^4}{x^6+m^2x^2} = \frac{mx^2}{x^4+m^2}.$$

If $m = 0$, then $f = 0/x^4 = 0 \to 0$ as $x \to 0$. If $m \neq 0$, then $f \to 0/m^2 = 0$ as $x \to 0$. Finally, along the vertical line $x = 0$, $f = 0/m^2 = 0 \to 0$ as $x \to 0$. Thus f goes to 0 along every line through the origin.

F2. Along the curve $y = x^3$,

$$f = \frac{x^6}{x^6+x^6} = 1/2.$$

Thus $f \to 1/2$ along this curve. Because this apparent limit disagrees with those found in part F1, we conclude that $\lim_{(x,y)\to(0,0)} f$ does not exist.

G. No, there does not exist a function $f(x, y)$ such that $\nabla f = \langle x^2 \cos(x^3y^3), y^2 \sin(x^3y^3)\rangle$. For suppose such an f did exist. Then

$$f_x = x^2 \cos(x^3y^3) \Rightarrow f_{xy} = -x^2 \sin(x^3 3y^2),$$

while

$$f_y = y^2 \sin(x^3y^3) \Rightarrow f_{yx} = y^2 \cos(3x^2y^3).$$

Because f_{xy} and f_{yx} are compositions of continuous functions, they are continuous, and by Clairaut's theorem they must agree. But they do not. Hence f cannot exist.

H. In Cartesian coordinates, the circle of radius R centered at $(R, 0)$ is

$$(x - R)^2 + y^2 = R^2.$$

Substituting $x = r\cos\theta$ and $y = r\sin\theta$ yields

$$(r\cos\theta - R)^2 + (r\sin\theta)^2 = R^2,$$

which is equivalent to

$$r^2\cos^2\theta - 2rR\cos\theta + R^2 + r^2\sin^2\theta = R^2,$$

which simplifies to

$$r - 2R\cos\theta = 0.$$