This exam begins for you when you open (or peek inside) this packet. It ends at 9:50 AM on Wednesday 2009 February 25. Between those two times, you may work on it as much as you like. I recommend that you get started early and work often. The exam is open-book and open-note, which means, precisely:

(1) You may freely consult all of this course's material: the Bretscher textbook, your class notes, your old homework and exam, and the materials on the course web site. If you missed a class and need to copy someone else's notes, do so before either of you begins the exam.

(2) You may assume all theorems discussed in class or in the assigned sections of the book. You do not have to prove or reprove them on this exam. On the other hand, you may not cite theorems that we have not studied. If you are unsure of whether you are allowed to cite a theorem, just ask.

(3) You may not consult any other papers, books, microfiche, film, video, audio recordings, Internet sites, etc. You may use a computer for these four purposes: viewing the course web site materials, running *Mathematica*, typing up your answers, and e-mailing with me. If you use *Mathematica*, then you may not load any packages. You may use a hand-held calculator instead of *Mathematica*, if you like. You may not share any of these materials with another student.

(4) **You may not discuss the exam in any way (spoken, written, pantomime, semaphore, etc.) with anyone but me until everyone has handed in the exam — even if *you* finish earlier.** During the exam you will inevitably see your classmates around campus. Please refrain from asking even seemingly innocuous questions such as "Have you started the exam yet?" If a statement or question conveys any information, then it is not allowed; if it conveys no information, then you have no reason to make it.

During the exam you may want to ask me questions. You may ask clarifying questions for free. If you believe that the statement of a problem is wrong, then you should certainly ask for clarification. You may also ask for hints, which cost you some points, to be decided by me as I grade your paper. I will not give you a hint unless you unambiguously request it. I will try to check my e-mail frequently, but there is always some lag and doing math over e-mail is not easy; you might want to talk to me in person.

Your solutions should be thorough, self-explanatory, and polished (concise, neat, and well-written, employing complete sentences with punctuation). Always show enough work so that a classmate could follow your solutions. Do not show scratch work, false starts, circuitous reasoning, etc. If you cannot solve a problem, write a *brief* summary of the approaches you've tried. Submit your solutions in a single stapled packet, presented in the order they were assigned.

Partial credit is often awarded. Exam grades are loosely curved — by this I do not mean that there are predetermined numbers of As, Bs, Cs to be awarded, but rather that there are no predetermined scores required for grades A, B, C.

Good luck!

Curve Fitting

You work at a company that develops wind turbines (not to solve our energy crisis — you just enjoy irritating birds). Your boss, Ms. Ogunmola, wants you to fit a curve of the form

$$y = a + b\cos x + c\cos^2 x + d\cos(2x)$$

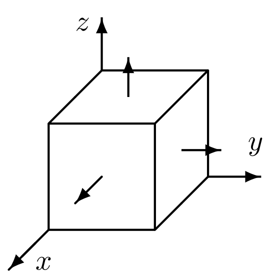
to some wind speed data $(x_1, y_1), \ldots, (x_N, y_N)$.

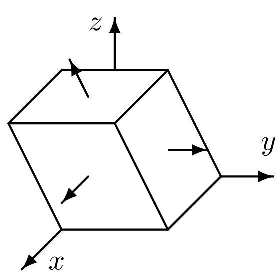
1. Explain in detail how you would find the fit curve using techniques of this course. (Warning: To check your answer, you might want to make up four data points and work out the solution explicitly.)

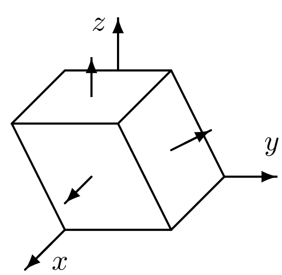
Transforming Normal Vectors

In 3D computer graphics applications such as video games, an object is described as a list of polygons. Each polygon is described by its vertices, which are points in $\mathbb{R}^3$. For a variety of reasons, it is also useful to give each polygon a *normal vector* $\vec{n}$ — meaning a nonzero vector, perpendicular to the polygon, that points "out" from the object. See the cube model below left.

Frequently (as in, millions of times per second) we wish to transform our polygons by a 3×3 matrix A. Transforming the vertices of a polygon is simple; we just apply A to each vertex. Transforming the normal vector is more subtle; if we just apply A to the normal vector $\vec{n}$, then the resulting vector $A\vec{n}$ may no longer be perpendicular to the polygon; see the transformed cube below in the middle. To produce transformed normals that are perpendicular to their transformed polygons, we must transform them in a different way; see below right.







2. Show that if $\vec{n}$ is perpendicular to a given polygon, then $\left(A^{-1}\right)^\top \vec{n}$ is perpendicular to the transformed polygon. That is, vertices transform by A but normal vectors transform by $\left(A^{-1}\right)^\top$.

3. What happens in the special case when A is a rotation? Explain in detail.

Properties Of The Trace

In these two problems all matrices are $n \times n$.

4. Using only the definitions of trace and matrix multiplication, prove that for any two matrices A and B,

$$\operatorname{tr}(AB) = \operatorname{tr}(BA).$$

5. Using Problem 4, prove that for any matrix C and any invertible matrix S,

$$\operatorname{tr}(SCS^{-1}) = \operatorname{tr} C.$$

Mirror Space

In this problem, V is any finite-dimensional vector space. Let n be its dimension. Let $\bar{V}$ denote the set of all linear transformations $f : V \to \mathbb{R}$. You should check for yourself that $\bar{V}$ is a vector space. (What are its operations? What is its 0 element?) We call $\bar{V}$ the *mirror space* of V.

Let $\mathcal{B} = \{v_1, \ldots, v_n\}$ be a basis for V. Then from $\mathcal{B}$ we can obtain a *mirror basis* $\bar{\mathcal{B}} = \{f_1, \ldots, f_n\}$ for the mirror space $\bar{V}$ by defining

$$f_i(v_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

6. Prove that $\bar{\mathcal{B}}$ is really a basis for $\bar{V}$.

Let $T : V \to V$ be any linear transformation. Then there is a *mirror transformation* $\bar{T} : \bar{V} \to \bar{V}$ defined by

$$(\bar{T}(f))(v) = f(T(v)).$$

You should check for yourself that $\bar{T}$ is really a linear transformation.

7. What is the relationship between $[T]_{\mathcal{B}}$ and $[\bar{T}]_{\bar{\mathcal{B}}}$?