Carleton College Math 232, Winter 2009, Exam 1

You have 70 minutes.

You may not use any notes or calculator.

Except on TRUE/FALSE/PUNT questions, always show your work and explain your answers. Good work often earns partial credit. A correct answer with no explanation often earns little or no credit.

If you have no idea how to solve a problem, or if you have forgotten a key formula that you think you need to know, then you may ask me for a hint. The hint will cost you some points (to be decided by me as I grade your paper), but will probably help you earn more points overall.

Good luck.

1. Every morning thousands of people drive from town T and village V to the city C to work, along the eastbound roads depicted in the map. Each road is labelled with its load (the number of cars on the road, on a typical morning, in thousands). Write a system of linear equations for the unknown loads. You do not need to solve the system.

2. Find all solutions of $\begin{bmatrix} 0 & 3 & 1 & 0 \\ 1 & 2 & -1 & 1 \\ 2 & 2 & 1 & 1 \\ -4 & 4 & -3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ -4 \\ 28 \end{bmatrix}$.

3 **A**. Let t be any real number, and let $A = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}$. Compute A^{-1}.

B. Explain A and A^{-1} geometrically, in words and pictures. Why does your answer to Part A make sense?

4. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation. Let L be any line in $\mathbb{R}^2$. The function f sends the line L to some set of points in $\mathbb{R}^2$; call this image set $f(L)$.

A. Assume that $\ker f = \left\{\vec{0}\right\}$. Prove that $f(L)$ is also a line.

B. What happens if you don't assume that $\ker f = \left\{\vec{0}\right\}$?

5. Let $\vec{u}$ be a unit vector in $\mathbb{R}^3$. Let L be the line containing $\vec{u}$ (and the origin). Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be the (orthogonal) projection of $\mathbb{R}^3$ onto this line.

A. What properties must f satisfy? (That is, how might you check your answer to Part B?)

B. Find the matrix for f, in terms of u_1, u_2, u_3. (If you cannot do this for arbitrary $\vec{u}$, then do some examples for partial credit.)

6. Each part A-H is a true/false question, but there are three valid answers: TRUE, FALSE, and PUNT. If you answer PUNT, then you receive half credit. Otherwise, if you answer correctly then you receive full credit, and if you answer incorrectly then you receive no credit. No explanation is necessary. Do not just write T, F, or P; write the entire word, clearly.

A. The rank of a matrix cannot exceed its number of rows.

B. For any two vectors $\vec{x}$, $\vec{y}$ in $\mathbb{R}^2$, any vector in $\mathbb{R}^2$ is a linear combination of $\vec{x}$ and $\vec{y}$.

C. (Let A be $p \times q$.) If the rank of A equals the number of columns in A, then for every $\vec{b}$ in $\mathbb{R}^p$ there is a unique solution to $A\vec{x} = \vec{b}$.

D. (Let A, B be $n \times n$.) If A and B are diagonal, then they must commute.

E. (Let A, B be $n \times n$.) If A and B are invertible, then $A + B$ must be invertible as well.

F. (Let A, B be $n \times n$.) If A and B are invertible, then AB must be invertible as well.

G. (Let A be $n \times n$.) A and A^2 must have the same image.

H. (Let A be $n \times n$.) A and A^2 must have the same kernel.