A Topological Problem Of Vector Fields

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1 The Problem

Let U be any open subset of $\mathbb{R}^3$. Recall that a *scalar field* on U is a smooth function $f : U \to \mathbb{R}$ and a *vector field* on U a smooth function $\vec{F} : U \to \mathbb{R}^3$. (By *smooth* we mean infinitely-differentiable.) The *gradient* of a scalar field f is the vector field

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right).$$

A vector field $\vec{F}$ is called a *gradient vector field* if it is a gradient — that is, if there exists some f such that $\vec{F} = \nabla f$. The *curl* of a vector field $\vec{F} = (P, Q, R)$ is the vector field

$$\operatorname{curl} \vec{F} = \left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}, \frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}, \frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} \right).$$

It is easy to check that

$$\operatorname{curl} \nabla f = \vec{0}$$

for any scalar field f. (Just compute it out and use the fact that mixed partials agree — e.g., $\frac{\partial}{\partial x} \frac{\partial}{\partial y} f = \frac{\partial}{\partial y} \frac{\partial}{\partial x} f$.) In other words, every gradient vector field has curl $\vec{0}$.

My problem is: Does the converse hold? Is every curl-$\vec{0}$ vector field a gradient?

2 Answers From Topology

The answer to the problem depends on which open set U we're talking about.

If U is all of $\mathbb{R}^3$, then it turns out that, yes, every curl-$\vec{0}$ vector field is a gradient.

For another example, let U be $\mathbb{R}^3$ with the z-axis deleted. Picture this U as $\mathbb{R}^3$ with an infinitely long hole drilled through it. It turns out that this hole allows for the existence of a curl-$\vec{0}$, non-gradient vector field $\vec{F}$. This vector field is essentially unique, in that all others are of the form $c\vec{F} + \nabla f$ for some constant c and scalar field f.

More generally — and here is where topology comes in — suppose that we bend the z-axis into some other smooth curve Z, perhaps by whacking it with a baseball bat it at the origin. Let U be $\mathbb{R}^3$ with this Z removed. As long as Z still goes "all the way through $\mathbb{R}^3$", the conclusion is unchanged; there is exactly one independent solution to the problem on U.

More generally still, suppose that we delete some number $k \geq 0$ of disjoint curves like this from $\mathbb{R}^3$; it turns out that there are exactly k independent solutions to the problem.

I am trying to convey to you a phenomenon that I think is rather remarkable. Finding an f whose gradient is a given $\vec{F} = (P, Q, R)$ amounts to solving three partial differential equations,

$$\frac{\partial f}{\partial x} = P, \qquad \frac{\partial f}{\partial y} = Q, \qquad \frac{\partial f}{\partial z} = R.$$

According to the claims above, finding the number of independent solutions boils down to counting the holes running through the space. A seemingly delicate problem of analysis and algebra ends up being merely topological!

The proofs of the claims in this section are not easy, but neither are they outlandishly difficult; they can be understood by any student with a solid knowledge of multivariable calculus, including Stokes' theorem and path-independence of gradient vector fields.

In differential topology (a branch of topology that uses calculus) all of these concepts are codified into a structure (a ring, in the jargon of abstract algebra) called *de Rham cohomology*. In the special case of $\mathbb{R}^3$ that we have been considering, de Rham cohomology reduces down to concrete questions of gradient, curl, and divergence of vector fields.

3 An Explicit Answer

Topology doesn't automatically tell us how to write down the solutions, but it does give us some ideas. Let's illustrate this in the case where U is $\mathbb{R}^3$ with the z-axis removed. We have to write down a vector field, but it doesn't need to be defined along the z-axis, and that's a golden opportunity. The z-axis is where both $x = 0$ and $y = 0$; in other words, it's where $x^2 + y^2 = 0$. This insight, a little intuition for curl, and some trial and error lead me to guess this solution:

$$\vec{F} = \left(\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2}, 0 \right).$$

You can easily check that curl $\vec{F} = \vec{0}$. In the remainder of this section, we explain (almost, but not quite, rigorously) why $\vec{F}$ cannot be a gradient. This calculus exercise builds valuable topological intuition.

For the sake of contradiction, suppose that $\vec{F} = \nabla f$ for some f. By the definition of the gradient, $\frac{\partial f}{\partial x} = \frac{-y}{x^2+y^2}$, so

$$f = \int \frac{-y}{x^2 + y^2}\, dx = -\arctan\left(\frac{x}{y}\right) + g(y, z)$$

for some $g(y, z)$. Then

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(-\arctan\left(\frac{x}{y}\right) + g(y, z) \right) = \frac{x}{x^2 + y^2} + \frac{\partial g}{\partial y}.$$

Since $\frac{\partial f}{\partial y} = \frac{x}{x^2+y^2}$, we have $\frac{\partial g}{\partial y} = 0$ and so $g(y,z) = g(z)$. Since $0 = \frac{\partial f}{\partial z} = \frac{dg}{dz}$, we have $g = c$, a constant. So the solution is

$$f = -\arctan\left(\frac{x}{y}\right) + c,$$

at least for $y \neq 0$. Can we somehow continue this solution across $y = 0$?

Here's an attempt. Begin with $\frac{\partial f}{\partial y} = \frac{x}{x^2+y^2}$ and follow the same method as above to obtain

$$f = \arctan\left(\frac{y}{x}\right) + c$$

for $x \neq 0$. This looks different from the first solution, until we realize that $-\arctan\left(\frac{x}{y}\right)$ and $\arctan\left(\frac{y}{x}\right)$ differ by a constant, as long as x and y are nonzero. Perhaps, by choosing the right constant, we can patch these two solutions together to get a single f that works on all of U?

Let's begin by declaring $f = -\arctan\left(\frac{x}{y}\right)$ for $y > 0$. In quadrant II, where $x < 0$ and $y > 0$,

$$-\arctan\left(\frac{x}{y}\right) = \arctan\left(\frac{y}{x}\right) + \pi/2.$$

The function on the right extends across $y = 0$ into quadrant III, where $x < 0$ and $y < 0$. Then, in quadrant III,

$$\arctan\left(\frac{y}{x}\right) + \pi/2 = -\arctan\left(\frac{x}{y}\right) + \pi.$$

The function on the right extends across $x = 0$ into quadrant IIII, where $x > 0$ and $y < 0$. In quadrant IIII,

$$-\arctan\left(\frac{x}{y}\right) + \pi = \arctan\left(\frac{y}{x}\right) + 3\pi/2.$$

Now, the function on the right extends into quadrant I, but it does not agree with our original choice of f there:

$$\arctan\left(\frac{y}{x}\right) + 3\pi/2 = -\arctan\left(\frac{x}{y}\right) + 2\pi \neq -\arctan\left(\frac{x}{y}\right).$$

So we have failed to find an f defined everywhere on U.

What happened? Essentially, arctan is measuring some angle, and taking a turn around the z-axis unavoidably changes this angle by 2π, so that when we return to where we started our function is off by 2π. (Students of complex analysis, this should remind you of the complex logarithm, with its "branch cut" issue.)

Is there another choice of f that would work? No. The two solutions (for $y \neq 0$ and $x \neq 0$) are unique up to a constant, and this constant is uniquely determined whenever we move from one quadrant to another. The only freedom we have in this process is the addition of an arbitrary constant to the initial f. This does not help; the constant gets added to f in every quadrant, and f is still off by 2π once it has gone around the circle. There is no consistent choice for f on all of U, and so $\vec{F}$ is not a gradient.