This exam begins for you when you open (or peek inside) this packet. It ends at 1:50 PM on Monday 2008 February 25. Between those two times, you may work on it as much as you like. I recommend that you get started early.

During the exam you may want to ask me questions. You may ask clarifying questions for free. If you believe that the statement of a problem is wrong, then you should certainly ask for clarification. You may also ask for hints, which cost you some points, to be decided unilaterally by me as I grade your paper. All hints are small; if you like, you can return for another hint. I will not give you a hint unless you unambiguously request it. For efficiency, try to have all questions formulated precisely before you ask them. I will try to check my e-mail frequently over the weekend, but there is always some lag, and doing math over e-mail is not easy.

The exam is open-book and open-note, which means, precisely:

- You may freely consult all of this class' material: the Munkres textbook, your class notes, your old homework and exam, and the materials on the class web site. If you missed a lecture and need to copy someone else's class notes, do so before either of you begins the exam.
- You may assume all results proved in class or in the assigned sections of the book. You do not have to prove or reprove them on this test. On the other hand, you may not cite results that we have not studied. If you are unsure of whether you are allowed to cite a result, just ask.
- You may not consult any other papers, books, microfiche, film, video, audio recordings, Internet sites, etc. You may use a computer for these three purposes: viewing the class web site materials, typing up your answers, and e-mailing with me. You may not use a computer or calculator for any other purpose.
- You may not discuss the exam in any way (spoken, written, pantomime, semaphore, etc.) with anyone but me until 1:50 PM on Monday 2008 February 25, even if you finish earlier. During the exam you will inevitably see your classmates around campus. Please refrain from asking even seemingly innocuous questions such as "Have you started the exam yet?" If a statement or question conveys any information, then it is not allowed; if it conveys no information, then you have no reason to make it.

Your solutions should be rigorous, self-explanatory, and polished (concise, neat, and well-written, employing complete sentences with punctuation). Always show enough work so that a classmate could follow your solutions. Do not show scratch work, false starts, circuitous reasoning, etc. If you cannot solve a problem, write a *brief* summary of the approaches you've tried. Submit your solutions in a single stapled packet, presented in the order they were assigned.

Partial credit is often awarded. Exam grades will be loosely curved — by this I do not mean that there are predetermined numbers of As, Bs, Cs to be awarded, but rather that there are no predetermined scores required for grades As, Bs, Cs.

Good luck!

1. Connectedness And Liftings

Part B of this problem is a claim that we used in our lifting arguments in class but did not prove. I'm asking you to prove it now.

A. Let X be a connected topological space. Let $\{U_1, \ldots, U_n\}$ be an open cover of X (with the U_i are nonempty and distinct). Suppose that the set $\{U_1, \ldots, U_n\}$ is partitioned into two nonempty disjoint sets $\{V_1, \ldots, V_m\}$, $\{W_1, \ldots, W_\ell\}$. Prove that there is some W_k that intersects the union $\bigcup_{j=1}^m V_j$.

B. Suppose that $p : E \to B$ is a covering space. Let X be compact and connected and $f : X \to B$ continuous. Show that $f(X) \subseteq B$ can be covered by finitely many evenly-covered open sets $U_1, \ldots, U_n \subseteq B$ such that, for all $k = 2, \ldots, n$,

$$f(X) \bigcap U_k \bigcap \left(\bigcup_{i=1}^{k-1} U_i \right) \neq \emptyset.$$

2. Covering Spaces

Recall that an *n-manifold* is a Hausdorff space in which each point has a neighborhood that is homeomorphic to an open subset of $\mathbb{R}^n$. (This is the definition we have always used in class; it appears to be slightly weaker than the definition given on page 225.) Let $p : E \to B$ be a covering space. For the following two questions, prove or give a counterexample.

A. If B is an n-manifold, then must E also be an n-manifold?

B. If B is simply connected and E is path-connected, then must E also be simply connected?

3. Malicious Art Curation

You're an assistant art curator in charge of hanging paintings in a museum. Here is a cartoon of a painting, attached to a cable at one point, hanging on a wall from a single nail:

(Of course the cable rests on the nail; I have left some space between them for clarity.) If the nail comes out of the wall, then the painting will crash to the floor. For safety, your boss insists that you hang each painting with *two* nails. She expects you to hang the painting like this:

Then, if either nail comes out of the wall, the painting will sag a bit but it will not fall all the way to the floor (which we assume to be far below the painting). However, you don't like your boss, and frankly you're pretty tired of paintings in general. You realize that by running the cable creatively around the two nails, you can make one of them useless. For example:

If the right nail comes out of the wall, then the painting sags but does not crash to the floor. If the left nail comes out, however, the cable slips around the right nail and the painting crashes.

A. In a cartoon like those above, show how to arrange the cable so that while both nails are in the wall the painting stays up, but if *either* nail comes out of the wall then the painting crashes to the floor. (The only freedom you have is in how the cable is arranged. You are not allowed to change how it is attached to the painting, you are not allowed to knock holes in the wall, etc.)

B. What does this have to do with our course? Equate the wall with $\mathbb{R}^2$, if you like.

4. Compact-Open Topology

For any topological spaces X and Y, let $C(X,Y)$ denote the set of continuous functions $f : X \to Y$. Endow $C(X,Y)$ with the *compact-open topology* described on pages 285-286 of your book. (When $X = \mathbb{R}$ this is similar to, but not quite the same as, Exam 1 #5.)

For the remainder of this problem, Y is any compact, Hausdorff topological space.

A. For any spaces X and Z, define a map

$$m : C(X,Y) \times C(Y,Z) \to C(X,Z)$$

by $m(f,g) = g \circ f$. Prove that m is continuous.

B. For any space Z, define a map $e : Y \times C(Y,Z) \to Z$ by $e(y,g) = g(y)$. Using part A — not some other method — prove that e is continuous. (Free hint: What if X were the one-point space $\{p\}$?)

C. Let G be the set of homeomorphisms from Y to Y. Then G is a group under composition. Also, G is a subset of $C(Y,Y)$, so it is a topological space, under the subspace topology from the compact-open topology on $C(Y,Y)$. Prove that G is a *topological group*, as defined on page 145.

5. Time Spent

Once you are done with the exam, please write the total time you spent on it, rounded to the nearest hour. Your answer does not affect your grade.