1. Here are six concepts: topology, basis, subbasis, metric, inner product, norm. Describe precisely how they relate to each other; which induces which, and how?

Answer: The concepts of inner product and norm exist only on vector spaces; the other concepts can be defined on any set.

- An inner product $\langle \cdot, \cdot \rangle$ induces a norm by $||v|| = \langle v, v \rangle^{1/2}$.
- A norm $|| \cdot ||$ induces a metric by $d(x, y) = ||x - y||$.
- A metric d induces a basis $\{B(x, \epsilon) : x \in X, \epsilon > 0\}$, where $B(x, \epsilon) = \{y \in X : d(x, y) < \epsilon\}$.
- A subbasis also induces a basis — the one consisting of all finite intersections of elements in the subbasis.
- A basis induces a topology consisting of all unions of elements in the basis.

2. Let X be any Hausdorff space. Prove that any one-point subset $\{x\}$ of X is closed.

Answer: Let $x \in X$. For any other $y \in X$, let U_y, V_y be disjoint open sets such that $x \in U_y$ and $y \in V_y$; these sets exist by the Hausdorff condition. Let $V = \cup_y V_y$. Since each V_y is open, so is V. Furthermore, V contains every $y \neq x$, and V does not contain x. Thus $V = X - \{x\}$ is open, so $\{x\}$ is closed.

3. Suppose Y is a subspace of X and A a subset of Y. Answer ONE of the following. Mark a giant X through the other one. There is no extra credit for answering both.

A. Is the closure of A in Y equal to the closure of A in X? Prove or give a counterexample.

Answer: They are not equal. Let $X = \mathbb{R}$, $Y = (0, 1)$, and $A = (0, 1)$. Then the closure of A in Y is $(0, 1)$, since $A = Y$ is closed in Y, but the closure of A in X is $[0, 1]$.

B. Is the interior of A in Y equal to the interior of A in X? Prove or give a counterexample.

Answer: They are not equal. Let $X = \mathbb{R}$, $Y = [0, 1]$, and $A = [0, 1]$. Then the interior of A in Y is $[0, 1]$, since $A = Y$ is open in Y, but the interior of A in X is $(0, 1)$.

4. Let $X = [-1, 1] \times (-1, 1) \subseteq \mathbb{R}^2$ in the subspace topology. Let $Y = [-1, 1) \times (-1, 1)$ as a subset of $\mathbb{R}^2$. Define $f : X \to Y$ by

$$f(x, y) = \begin{cases} (x, y) & \text{if } x \neq 1, \\ (-1, -y) & \text{if } x = 1. \end{cases}$$

Endow Y with the quotient topology from f. In words and/or pictures, describe Y as a space. Describe its open sets. Is it a manifold? (Your answers to this problem need not be rigorous, but try to explain as well as you can.)

Answer: In the quotient, the left- and right-hand edges of X (where $x = \pm 1$) are glued together, but in an upside-down fashion (due to the $-y$). The result is a Möbius strip.

There are two kinds of open subsets in Y. The first kind consists of those that do not intersect $\{-1\} \times (-1, 1)$; these correspond one-to-one with open subsets of X that do not touch the edges. The second kind consists of those that contain an interval of the form $\{-1\} \times (a, b)$. Whenever an open set contains $\{-1\} \times (a, b)$, it must also contain an open set of points near $\{1\} \times (-b, -a)$. [This is most easily explained in a picture, which I'll omit in this PDF; talk to me if you have trouble understanding.]

Yes, Y is a manifold. In it, the left- and right-hand edges of X are glued seamlessly so that there is no longer any edge at all. Every point in Y possesses a neighborhood homeomorphic to an open disk in $\mathbb{R}^2$. For points (x, y) with $x \neq -1$, the disk looks like an ordinary disk in $\mathbb{R}^2$. For points $(-1, y)$, the disk contains material on the left- and right-hand sides of Y. [Again this is most easily explained in a picture.]

5. Let Y be any topological space. Let F be the set of all continuous functions $f : \mathbb{R} \to Y$. For any closed interval $C \subseteq \mathbb{R}$ and open $U \subseteq Y$, let

$$S(C, U) = \{f : f(C) \subseteq U\} \subseteq F.$$

Let T be the topology on F generated by all of these subsets $S(C, U) \subseteq F$. Finally, define a function $e : \mathbb{R} \times F \to Y$ by $e(x, f) = f(x)$. Prove that e is continuous.

Answer: Let U be open in Y. Let $(x, f) \in e^{-1}(U)$. That is, f is continuous and $f(x) \in U$. Thus $f^{-1}(U)$ is open and $x \in f^{-1}(U)$. It follows that $f^{-1}(U)$ contains $(x - \epsilon, x + \epsilon)$ for some $\epsilon > 0$. Let

$$V = \left(x - \frac{\epsilon}{2}, x + \frac{\epsilon}{2}\right) \times S\left(\left[x - \frac{\epsilon}{2}, x + \frac{\epsilon}{2}\right], U\right).$$

Then (x, f) is an element of V, since $x \in (x - \frac{\epsilon}{2}, x + \frac{\epsilon}{2})$ and f is a continuous function that sends $\left[x - \frac{\epsilon}{2}, x + \frac{\epsilon}{2}\right]$ into U. Also, V is a basis element for $\mathbb{R} \times F$ and $e(V) \subseteq U$. In summary, V is an open neighborhood of (x, f) such that $V \subseteq e^{-1}(U)$. Since any $(x, f) \in e^{-1}(U)$ possesses an open neighborhood contained in $e^{-1}(U)$, it follows that $e^{-1}(U)$ is open. Since this is true for all U open in Y, the map e is continuous.

Remark: In this problem I used closed intervals C for the sake of simplicity. If instead one uses closed, bounded (that is, compact) subsets $C \subseteq \mathbb{R}$, then the resulting topology on F is called the *compact-open topology*. This topology is useful in a variety topological constructions.