1. Find the equation of the tangent line to $y = 4x^3 - 3x$ at $x = 2$.

Answer: First, $y(2) = 4 \cdot 8 - 3 \cdot 2 = 26$. Second, $y' = 12x^2 - 3$, so $y'(2) = 12 \cdot 4 - 3 = 45$. The tangent line is then $y - 26 = 45(x - 2)$, or $y = 45x - 64$.

Remark: This problem is similar to several assigned homework problems, such as 3.1 #36, 3.2 #33, and 3.3 #24.

2. Find the acceleration of a particle, given that the particle's position is

$$s(t) = 2\sin(kt) - 9.8\sqrt{t} + 3t^2.$$

Answer: The acceleration is the second derivative $s''(t)$.

$$s'(t) = 2\cos(kt)k - 4.9t^{-1/2} + 6t$$
$$\Rightarrow \qquad s''(t) = -2\sin(kt)k^2 + 2.45t^{-3/2} + 6.$$

Remark: This is similar to homework problem 3.1 #50.

3. Differentiate the following functions. Do NOT simplify.
A. $f(u) = \left(\dfrac{1}{2}\right)^{\cos u}$.

Answer:
$$f'(u) = \left(\frac{1}{2}\right)^{\cos u} \cdot \ln\left(\frac{1}{2}\right) \cdot (-\sin u).$$

B. $r = e^{1.3t}\left(t^{1.3} + t\right)$.

Answer:
$$\frac{dr}{dt} = e^{1.3t} \cdot 1.3 \cdot \left(t^{1.3} + t\right) + \left(e^{1.3t}\right) \cdot \left(1.3t^{0.3} + 1\right).$$

4. The table below shows some data. I wish to model y as a function of x. When I plot the data on a log-log plot, I get a line of slope 1.2 and intercept 2.7. What then is the function $y(x)$? Simplify your answer.

x	3	4	5	6	7
y	56	79	103	129	155

Answer: The equation of the line is $\ln y = 1.2\ln x + 2.7$, which implies that

$$y = e^{1.2\ln x + 2.7} = e^{2.7}e^{1.2\ln x} = e^{2.7}e^{\ln(x^{1.2})} = e^{2.7}x^{1.2}.$$

Remark: This is similar to homework problem 1.2 #26a and questions on exams from earlier terms.

5. Although it may have slipped your mind, you recently piloted an oil tanker from Anchorage, Alaska to Acapulco, Mexico. You measured the following speeds over the course of a single 24-hour period. How far did you travel during that period? Give a range of possible distances.

time	7 AM	10 AM	2 PM	4 PM	5 PM	9 PM	11 PM	7 AM
speed (km/h)	38	42	47	56	52	54	56	44

Answer: To estimate the distance travelled, we can assume constant speed on each given time interval, multiply by the length of that time interval to get the distance travelled, and add up those distances travelled. For my first estimate, I use the lesser of the two speeds measured at the ends of each time interval:

$$3 \cdot 38 + 4 \cdot 42 + 2 \cdot 47 + 1 \cdot 52 + 4 \cdot 52 + 2 \cdot 54 + 8 \cdot 44 = 1096.$$
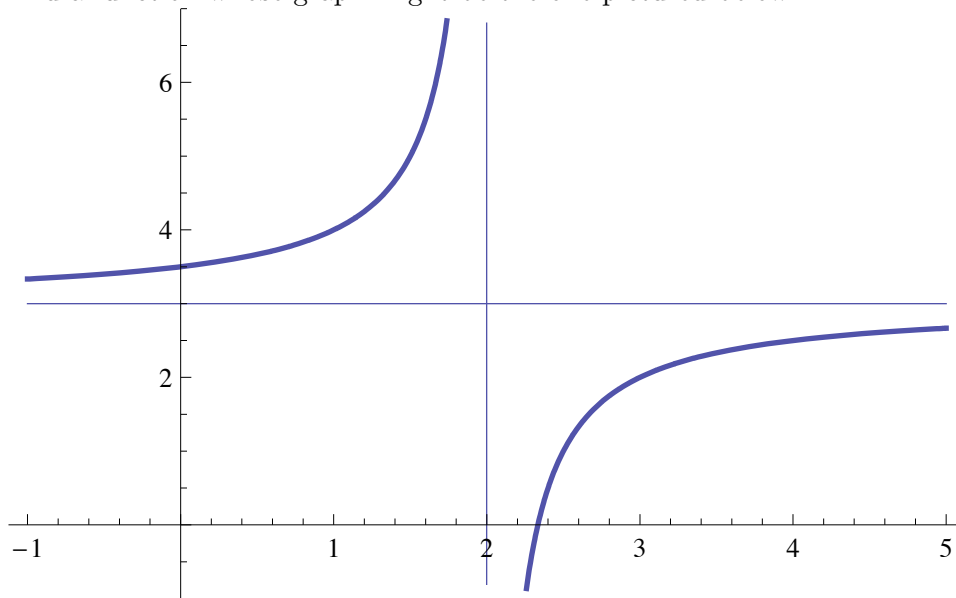
For my second estimate, I use the greater of the two speeds measured:

$$3 \cdot 42 + 4 \cdot 47 + 2 \cdot 56 + 1 \cdot 56 + 4 \cdot 54 + 2 \cdot 56 + 8 \cdot 56 = 1258.$$

The distance travelled seems to be between 1096 and 1258 km.

Remark: This is similar to homework problem 5.1 #14.

6. Find a function whose graph might be the one pictured below.

Answer: The given graph looks like the hyperbola $y = 1/x$. However, it is upside-down, so $y = -1/x$ is a better guess. It is also centered at $(2, 3)$ rather than $(0, 0)$, so

$$y - 3 = \frac{-1}{x - 2}$$

is a still-better guess. Our final answer is

$$y = \frac{-1}{x-2} + 3.$$

7. Consider the differential equation $\frac{dp}{dt} = mp$, where m is a constant.

A. What practical situations does this equation describe?

   Answer: The equation is used to describe any quantity p that grows or decays at a "constant relative rate" (meaning, a rate proportional to the quantity itself) with respect to time t. Examples include continuously compounding interest, population growth, radioactive decay, and decay of a drug as it is metabolized by the body.

B. What are the solutions of this equation? Convince me that your answer is correct.

   Answer: The general solution to $\frac{dp}{dt} = mp$ is

$$p = p_0 e^{mt},$$

where p_0 is an arbitrary constant. That this solution works is easily checked:

$$\Rightarrow \qquad \begin{aligned} p &= p_0 e^{mt} \\ \frac{dp}{dt} &= p_0 e^{mt} m \\ &= mp_0 e^{mt} \\ &= mp. \end{aligned}$$

8. From the definition of the derivative, compute $\frac{d}{dx} \cos(5x)$. Show all steps.

   Answer: By the definition of the derivative,

$$\begin{aligned} \frac{d}{dx}\cos(5x) &= \lim_{h \to 0} \frac{\cos(5(x+h)) - \cos(5x)}{h} \\ &= \lim_{h \to 0} \frac{\cos(5x + 5h) - \cos(5x)}{h} \\ &= \lim_{h \to 0} \frac{\cos(5x)\cos(5h) - \sin(5x)\sin(5h) - \cos(5x)}{h} \\ &= \lim_{h \to 0} \frac{\cos(5x)\left(\cos(5h) - 1\right) - \sin(5x)\sin(5h)}{h} \\ &= \lim_{h \to 0} \cos(5x)\frac{\cos(5h) - 1}{h} - \sin(5x)\frac{\sin(5h)}{h} \\ &= \lim_{h \to 0}\left(\cos(5x)\frac{\cos(5h) - 1}{h}\right) - \lim_{h \to 0}\left(\sin(5x)\frac{\sin(5h)}{h}\right) \\ &= \cos(5x)\lim_{h \to 0}\left(\frac{\cos(5h) - 1}{h}\right) - \sin(5x)\lim_{h \to 0}\left(\frac{\sin(5h)}{h}\right). \end{aligned}$$

Recall that

$$\lim_{u \to 0} \frac{\sin u}{u} = 1, \qquad\qquad \lim_{u \to 0} \frac{\cos u - 1}{u} = 0.$$

Let $u = 5h$; then

$$\cos(5x) \lim_{h \to 0} \left(\frac{\cos(5h) - 1}{h} \right) - \sin(5x) \lim_{h \to 0} \left(\frac{\sin(5h)}{h} \right)$$

$$= \cos(5x) \cdot 5 \cdot \lim_{h \to 0} \left(\frac{\cos(5h) - 1}{5h} \right) - \sin(5x) \cdot 5 \cdot \lim_{h \to 0} \left(\frac{\sin(5h)}{5h} \right)$$

$$= \cos(5x) \cdot 5 \cdot \lim_{5h \to 0} \left(\frac{\cos(5h) - 1}{5h} \right) - \sin(5x) \cdot 5 \cdot \lim_{5h \to 0} \left(\frac{\sin(5h)}{5h} \right)$$

$$= \cos(5x) \cdot 5 \cdot \lim_{u \to 0} \left(\frac{\cos(u) - 1}{u} \right) - \sin(5x) \cdot 5 \cdot \lim_{u \to 0} \left(\frac{\sin(u)}{u} \right)$$

$$= \cos(5x) \cdot 5 \cdot 0 - \sin(5x) \cdot 5 \cdot 1$$

$$= -\sin(5x) \cdot 5.$$

Therefore $\frac{d}{dx} \cos(5x) = -\sin(5x) \cdot 5$.

Remark: This is a slightly harder version of homework problem 3.3 #20.

9. You are given four points (x_0, y_0), (x_1, y_1), (x_2, y_2), and (x_3, y_3). You'd like to find a cubic curve $y = ax^3 + bx^2 + cx + d$ that is tangent at (x_0, y_0) to the line through (x_0, y_0) and (x_1, y_1), and tangent at (x_3, y_3) to the line through (x_2, y_2) and (x_3, y_3). (This problem is very common in computer graphics; for example, it arises when drawing *the letters you are reading right now*.)

A. Write a system of equations that can be used to find the cubic. Do not solve them.

Answer: The desired cubic $y = ax^3 + bx^2 + cx + d$ passes through (x_0, y_0), so

$$y_0 = ax_0^3 + bx_0^2 + cx_0 + d.$$

It also passes through (x_3, y_3), so

$$y_3 = ax_3^3 + bx_3^2 + cx_3 + d.$$

The derivative of the cubic is $y'(x) = 3ax^2 + 2bx + c$. At x_0 the derivative must equal the slope of the line through (x_0, y_0) and (x_1, y_1), which is $(y_1 - y_0)/(x_1 - x_0)$; thus

$$\frac{y_1 - y_0}{x_1 - x_0} = 3ax_0^2 + 2bx_0 + c.$$

Similarly, the tangency condition at x_3 implies that

$$\frac{y_3 - y_2}{x_3 - x_2} = 3ax_3^2 + 2bx_3 + c.$$

These four equations can be used to find a, b, c, and d, and hence the cubic.

B. Find the cubic for $(x_0, y_0) = (0, 0)$, $(x_1, y_1) = (1, 0)$, $(x_2, y_2) = (2, 1)$, $(x_3, y_3) = (3, 1)$.

Answer: Plugging the given (x_i, y_i) into the four equations from Part A yields

$$
\begin{aligned}
0 &= 0a + 0b + 0c + d, \\
1 &= 27a + 9b + 3c + d, \\
0 &= 0a + 0b + c, \\
0 &= 27a + 6b + c.
\end{aligned}
$$

We immediately see that $c = d = 0$. The equations simplify to

$$
\begin{aligned}
1 &= 27a + 9b, \\
0 &= 27a + 6b.
\end{aligned}
$$

The solution is $b = 1/3$ and $a = -2/27$. [I'll omit that work here.] Hence the cubic is

$$
y = \frac{-2}{27}x^3 + \frac{1}{3}x^2.
$$

Remark: This is a slightly easier version of homework problem 3.1 #72. It is essentially identical to 3.1 #65, which was not assigned.