Name:

I have adhered to the Duke Community Standard.

Signature:

Math 31L 03-04 Fall 2006 Exam 3

Instructions: You have 75 minutes. You may use your TI-83 or equivalent calculator. Always show all of your work. Partial credit is often awarded. Pictures are often helpful. Give simplified answers, as exact as possible. Put a box around each answer. Ask questions if any problem is unclear. Good luck.

1. Compute the following integrals. Give exact, simplified answers.

A. $\int_1^2 t^2 - 3t + 6\ dt$

Solution: [Use FTC1 to get 23/6.]

B. $\int_0^{\pi/4} \frac{1}{1+x^2}\ dx$

Solution: [Use FTC1 to get $\arctan(\pi/4) - \arctan(0) = \arctan(\pi/4)$. This is the simplified, exact answer I wanted.]

2. Let $f(x) = \int_1^x (t-3)^2(e^t - 1)\ dt$. Find all local maxima and minima of $f(x)$. Show your work.

Solution: [Differentiate using FTC2 to get $f'(x) = (x-3)^2(e^x - 1)$. So $f'(x)$ is never undefined; it is zero at $x = 3$ and $x = 0$. If you graph $y = (x-3)^2(e^x - 1)$ on your calculator, you see that $f'(x)$ is positive both just before and just after $x = 3$. Therefore $x = 3$ is an inflection point, not a minimum or maximum. Meanwhile, at $x = 0$, $f'(x)$ changes from negative to positive, so $x = 0$ is a local minimum. To finish the problem, we would like to find the value $f(0)$, if only approximately. Now

$$f(0) = \int_1^0 (t-3)^2(e^t - 1)\ dt.$$

In other words, it's the negative of the area under the graph of $f'(x)$ between $x = 0$ and $x = 1$. This area is roughly triangular, with base 1 and height

$$f'(1) = (1-3)^2(e^1 - 1) = 4(e-1) \approx 6.873.$$

So

$$f(0) \approx -1 \cdot 6.873/.2 = -3.4365.$$

3. The city of Superville, Nebraska covers a circular region of radius 4 km. At a distance of x km from the city center, the population density is about

$$m(x) = \frac{30}{\sqrt{x}} \text{ people/km}^2.$$

A. Where is the population most dense? Where is it least dense?

Solution: The function $m(x)$ is decreasing. The population is most dense at (near) the city center, $x = 0$. It's least dense on the boundary, $x = 4$.

B. Using n concentric rings, write a Riemann sum that approximates the total population of Superville.

Solution: Dividing the radius 4 into n parts produces $\Delta x = \frac{4}{n}$. The outer radius of the kth concentric ring is $x_k = k\frac{4}{n}$. The area of the kth ring is about

$$2\pi x_k \, \Delta x = 2\pi k \frac{4}{n}\frac{4}{n}.$$

The number of people in that ring is about

$$m(x_k) \, 2\pi x_k \, \Delta x = \frac{30}{\sqrt{k\frac{4}{n}}} \cdot 2\pi k \frac{4}{n}\frac{4}{n}.$$

So the total population is about

$$\sum_{k=1}^{n} m(x_k) \, 2\pi x_k \, \Delta x = \sum_{k=1}^{n} \frac{30}{\sqrt{k\frac{4}{n}}} \cdot 2\pi k \frac{4}{n}\frac{4}{n}.$$

C. Write an integral that represents the total population.

Solution: The total population is the limit of the above sum as $n \to \infty$. So it's the integral

$$\int_0^4 m(x) \, 2\pi x \, dx.$$

D. Evaluate that integral exactly.

Solution: [Use FTC1 to get 320π.]

E. What is the average population density of Superville, in people per km^2?

Solution: The area is $\pi 4^2 = 16\pi$. So the average population density is $(320\pi)/(16\pi) = 20$.

4. You are the prime minister of Superland, a nation with a constant population of M people, some of whom are unfortunately infected with the Bad virus. Let $I(t)$ be the number of people with the virus on day t. The virus is mutating, which is causing its "infectiousness" (the ability to infect an exposed person) to change. Suppose that the rate of infection, in people per day, equals 10^{-7} times the product of the following three factors:

1. the number I of infected people,

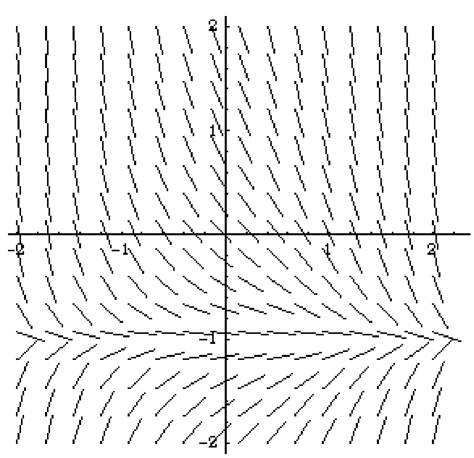
2. the number of people not infected, and

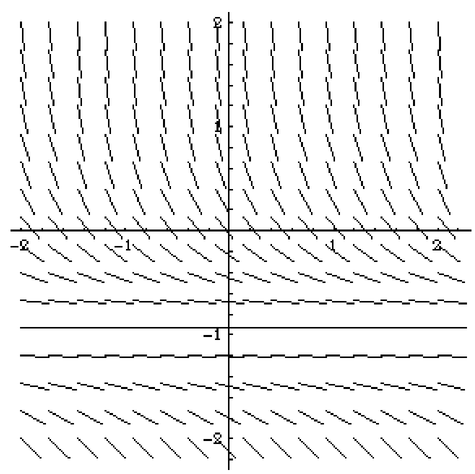
3. the infectiousness, which your best scientists say is $10^{-6}(I - M/2)$.

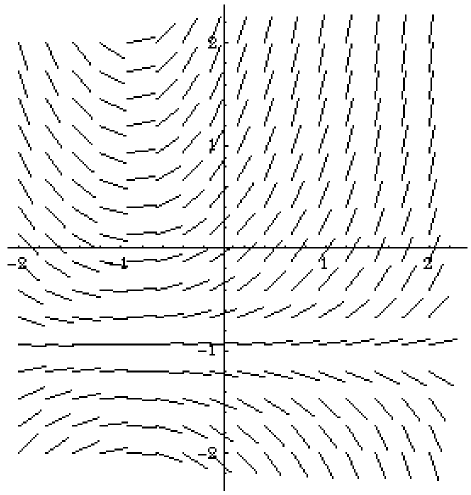
A. Write a differential equation for the number of infected people.
Solution: $dI/dt = 10^{-13}I(M - I)(I - M/2)$.
B. Find the equilibrium solutions of the differential equation.
Solution: They occur where $dI/dt = 0$, so at $I = 0$, $I = M$, $I = M/2$.
C. After making a quick survey of the country, your health minister guesses that *roughly* half of the population is infected. From this, can you determine the long-term behavior of the infection? If so, what is it? If not, why not? (Hint: You may want to sketch a slope field.)
Solution: The $I = M/2$ equilibrium is unstable, so initial conditions near it are repelled from it, either to $I = 0$ or $I = M$. So you need more precise information from your health minister before coming to any conclusions.
5. Label each slope field below with the letter (A, B, C, D) of the corresponding differential equation.

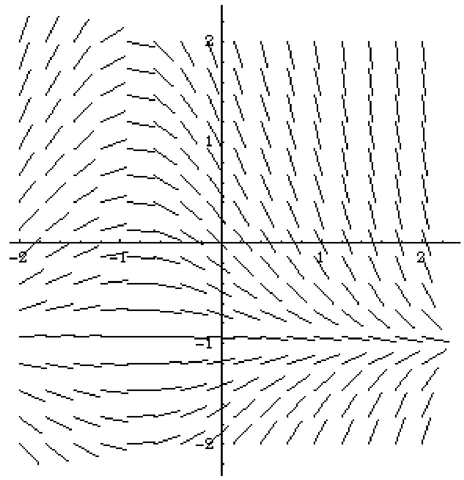
$$\text{A} : \frac{dy}{dx} = (-y - 1)(1 + x) \qquad \text{B} : \frac{dy}{dx} = (-y - 1)(1 + y)$$

$$\text{C} : \frac{dy}{dx} = (-y - 1)(1 + x^2) \qquad \text{D} : \frac{dy}{dx} = (-y - 1)(-1 - x)$$









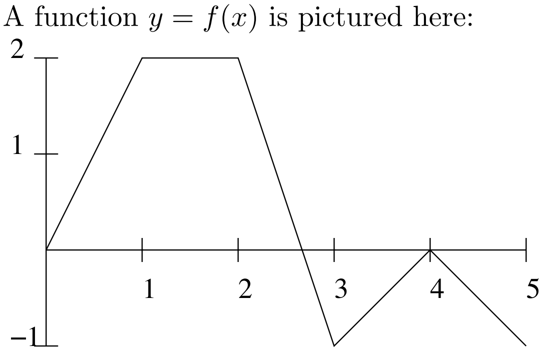
Solution: C is upper left, B is upper right, D is lower left, A is lower right.

6. Clearly and thoroughly describe the distinction between

$$\int f(x)\,dx \quad \text{and} \quad \int_a^b f(x)\,dx.$$

Solution: On the left, $\int f(x)\,dx$ is the antiderivative of $f(x)$, or rather all of them; it is a *class of functions*, namely all of the functions whose derivatives are $f(x)$. On the right, $\int_a^b f(x)\,dx$ is the integral of $f(x)$ from $x = a$ to $x = b$; it is a *number* expressing the signed area of the region of the plane trapped by $y = f(x)$, $y = 0$, $x = a$, and $x = b$.

7. A function $y = f(x)$ is pictured here:



A. On the picture, draw the area corresponding to the Riemann sum

$$\sum_{k=0}^{5} f\left(\frac{x_k + x_{k+1}}{2}\right)\Delta x,$$

where $\Delta x = 0.5$ and $x_k = 1 + k\Delta x$.

Solution: First, the sum has k running from 0 to 5, so there are 6 terms in the sum, so $n = 6$. Second, $x_0 = 1$ and $x_6 = 4$, so the sum is an approximation of an integral from 1 to 4. Third, it is a midpoint sum. [Now, on the picture, divide the interval from 1 to 4 into parts of size $\Delta x = 0.5$. Over/under each part, draw the rectangle that touches the graph $y = f(x)$ at the midpoint of that part.]

B. Let $F(x)$ be an antiderivative of $f(x)$. There are six pairs of numbers below. Between each pair, mark "<", ">", or "=" to indicate that relationship, or mark "?" to indicate that the relationship cannot be determined.

$$\int_0^5 f(x)\,dx \quad > \quad 0 \qquad\qquad \int_0^3 f(x)\,dx \quad < \quad 4$$

$$f'(0.5) \quad < \quad F(2) - F(0) \qquad\qquad F(3) - F(1) \quad > \quad 2$$

$$F(2) \quad ? \quad 2 \qquad\qquad F(4) \quad < \quad F(3)$$

8. Does every continuous function have an antiderivative? Explain.

Solution: Yes. Let $f(x)$ be any continuous function. Define $F(x) = \int_0^x f(t)\,dt$. Then FTC2 says that the derivative of $F(x)$ is $f(x)$. So we have demonstrated an antiderivative for $f(x)$.

9. One fine winter day, the temperature of Lake Mendota is 7° (centigrade) while the ambient air temperature is -10°. Let $x(t)$ be the temperature of the lake on day t afterwards. Assume that the air stays at -10°, and that the lake cools at a rate equal to twice the difference in temperature between it and the air.

A. Write a differential equation and an initial condition to describe this problem.

Solution: $dx/dt = -2(x + 10)$, with $x(0) = 7$.

B. Solve this initial value problem using separation of variables.

Solution: [Use separation to get $x = Be^{-2t} - 10$. Then the initial condition implies that $B = 17$. So $x(t) = 17e^{-2t} - 10$.]