Math 206-01, Spring 2007, Exam 1 Answers

1. Introduction

Nothing to see here; move along.

2. Bézier Splines

A. By differentiation and plugging in, $\vec{\alpha}(0) = \vec{a}_0$, $\vec{\alpha}(1) = \vec{a}_3$, $\vec{\alpha}'(0) = 3(\vec{a}_1 - \vec{a}_0)$, $\vec{\alpha}'(1) = 3(\vec{a}_3 - \vec{a}_2)$. So the curve should be drawn starting off from $\vec{a}_0$ going toward $\vec{a}_1$, and ending up at $\vec{a}_3$ from the direction of $\vec{a}_2$. (For typical choices of the control points, the curve will not go through either $\vec{a}_1$ or $\vec{a}_2$.)

B. For the two points to coincide we need $\vec{a}_3 = \vec{b}_0$. For the tangents to match we need $3(\vec{a}_3 - \vec{a}_2) = 3(\vec{b}_1 - \vec{b}_0)$; this means that the three points $\vec{a}_2$, $\vec{a}_3 = \vec{b}_0$, and $\vec{b}_1$ lie along a single line and are equidistant.

3. Zero Curvature

A. After we plug in $k = 0$ and $\tau = 1$, the equations become

$$\begin{aligned} \vec{t}' &= \vec{0}, \\ \vec{n}' &= -\vec{b}, \\ \vec{b}' &= \vec{n}. \end{aligned}$$

Comparing $\vec{n}$ and $\vec{b}$, we see that

$$\begin{aligned} \vec{t}' &= \vec{0}, \\ \vec{n}'' &= -\vec{n}, \\ \vec{b}'' &= -\vec{b}. \end{aligned}$$

B. These work:

$$\begin{aligned} \vec{t} &= (0, 0, 1), \\ \vec{n} &= (\cos t, -\sin t, 0), \\ \vec{b} &= (\sin t, \cos t, 0). \end{aligned}$$

C. Just antidifferentiate $\vec{t}$ to get $\vec{\alpha}(t) = (0, 0, t)$. (This assumes that the curve starts at the origin; that's not important.) The curve is a straight line. As t moves from $t = 0$ to $t = 2\pi$, the frame "twists" around the line once.

D. The torsion has no effect on the curve itself. Specifying $\tau = 0$ leads to the same line, but with the Frenet frame constant along it, not rotating.

What is a curve, really? Here are four traits that one might include in the definition.

(1) A curve is a *subset* of $\mathbb{R}^3$. It's not just any subset; it's "one-dimensional", like a line, but it's probably bent, not straight like a line.
(2) A curve might come equipped with an *orientation* — a notion that one direction is "forward" and the opposite direction is "backward".
(3) A curve might come with a *parametrization*, which explicitly describes how to draw the curve, or gives a sense of "time" to it.

(4) A curve might come with an *orthonormal frame* that moves along with the parametrization and describes how space twists near the curve.

We tend to focus on the third item, the parametrization. Specifying a parametrization automatically specifies the curve as a subset of $\mathbb{R}^3$. If the parametrization has nonvanishing speed, then we get the orientation for free. If the parametrization also has nonvanishing curvature, then we get the orthonormal frame for free as well. However, if the curvature vanishes, then the parametrization does not determine a unique orthonormal frame; that's extra information, to be specified separately from the parametrization.

4. Riparian Erosion

A. In our homework we did a modified Section 1.5 Problem 12 that computed $k\vec{n}$. Just take that formula, plug in $\vec{\alpha} = (x, y, 0)$, and compute.

B. The quantity $\partial\vec{\alpha}/\partial s$ measures the rate of change of the course of the river — that is, the rate of erosion. Comparing to Part A, we see that the problem is asking why the rate of erosion might equal $-|\vec{\alpha}'|^4 k\vec{n}$. Well, $-k\vec{n}$ is a vector pointing toward the outside of the curve; in the curved parts of the river it's large, while in the straight parts of the river it's small. The scaling factor $|\vec{\alpha}'|^4$ might arise because the rate of erosion increases "dramatically" as the speed $|\vec{\alpha}'|$ of the water increases.

C. Cut off x and y after the quadratic terms. Plug in $s = 0$ and set $x(t, 0) = t$ and $y(t, 0) = t^2$ to obtain $a_{00} = a_{20} = 0$, $a_{10} = 1$, $b_{00} = b_{10} = 0$, $b_{20} = 1$, so that

$$\begin{aligned} x(t,s) &= t + a_{01}s + a_{11}ts + a_{02}s^2, \\ y(t,s) &= b_{01}s + t^2 + b_{11}ts + b_{02}s^2. \end{aligned}$$

Differentiating once or twice and doing some algebra turns the differential equations into

$$\begin{aligned} \partial x/\partial s &= 4t + 2b_{11}s + 4a_{11}ts + 2a_{11}b_{11}s^2, \\ \partial y/\partial s &= -2 - 4a_{11}s - 2a_{11}^2 s^2. \end{aligned}$$

D. Antidifferentiate $\partial y/\partial s$ with respect to s. Terms without s may arise, but they need to equal t^2, so we get

$$y(t,s) = t^2 - 2s - 2a_{11}s^2 - \frac{2}{3}a_{11}^2 s^3.$$

Notice that the there is no ts term, so $b_{11} = 0$. Antidifferentiate $\partial x/\partial s$, make the t terms equal t, and use $b_{11} = 0$ to get

$$x(t,s) = t + 4ts + 2a_{11}ts^2.$$

Notice that the ts term has coefficient 4, which forces $a_{11} = 4$. However, I did not see it when I wrote the test; that's why I tell you to set $a_{11} = 1$ in the next part. My grading in these parts will be flexible.

E. Setting $a_{11} = 1$ (the $a_{11} = 4$ correction mentioned in the preceding part produces similar results, but I'll go ahead with the question as stated) and setting $s = 0.1$ produces

$$\begin{aligned} x(t,s) &= 1.42t, \\ y(t,s) &= t^2 - 0.220\bar{6}. \end{aligned}$$

F. Draw the parabola $y = x^2$ for the river at $s = 0$. The river at time $s = 0.1$ is also a parabola. I'll use the $a_{11} = 1$ choice as described above. Then it hits the y-axis when $t = 0$, so at $(x, y) = (0, -0.220\bar{6})$. Since $0.220\bar{6}$ is about $1/4$, it hits the x-axis when t is about $\pm 1/2$, so at about $(\pm 0.71, 0)$.

5. Stereographic Projection

Recall $\vec{x} : \mathbb{R}^2 \to S$ given by

$$\vec{x}(u, v) = \left(\frac{2u}{u^2 + v^2 + 1}, \frac{2v}{u^2 + v^2 + 1}, \frac{u^2 + v^2 - 1}{u^2 + v^2 + 1} \right).$$

As a lemma, compute $d\vec{x}_{(u,v)}$ and then multiply matrices to get

$$d\vec{x}_{(u,v)}^{\top} d\vec{x}_{(u,v)} = \frac{4}{(u^2 + v^2 + 1)^2} I,$$

where I is the 2×2 identity matrix; thus

$$(d\vec{x}_{(u,v)}\vec{q}) \cdot (d\vec{x}_{(u,v)}\vec{r}) = \frac{4}{(u^2 + v^2 + 1)^2} \vec{q} \cdot \vec{r}.$$

A. By the lemma,

$$|d\vec{x}_{(u,v)}\vec{q}| = \sqrt{d\vec{x}_{(u,v)}\vec{q} \cdot d\vec{x}_{(u,v)}\vec{q}} = \sqrt{\frac{4}{(u^2 + v^2 + 1)^2} \vec{q} \cdot \vec{q}} = \frac{2}{u^2 + v^2 + 1} |\vec{q}|.$$

B. Along the unit circle $u^2 + v^2 = 1$ (and only there) we have $|d\vec{x}_{(u,v)}\vec{q}| = |\vec{q}|$.

C. Let θ be the angle between $\vec{q}$ and $\vec{r}$ and ϕ the angle between $d\vec{x}_{(u,v)}\vec{q}$ and $d\vec{x}_{(u,v)}\vec{r}$. Then by the lemma and Part A,

$$\begin{aligned}
\cos \phi &= \frac{(d\vec{x}_{(u,v)}\vec{q}) \cdot (d\vec{x}_{(u,v)}\vec{r})}{|d\vec{x}_{(u,v)}\vec{q}||d\vec{x}_{(u,v)}\vec{r}|} \\
&= \frac{\frac{4}{(u^2+v^2+1)^2} \vec{q} \cdot \vec{r}}{\frac{2}{u^2+v^2+1}|\vec{q}| \frac{2}{u^2+v^2+1}|\vec{r}|} \\
&= \frac{\vec{q} \cdot \vec{r}}{|\vec{q}||\vec{r}|} \\
&= \cos \theta.
\end{aligned}$$

Since both angles are in $[0, \pi]$, we conclude that $\theta = \phi$.

D. Parametrize the circle C by $\vec{\alpha}(t) = (1 + \cos t, \sin t)$. Plug these in for u and v to get $\vec{x} \circ \vec{\alpha}(t)$; call that composition $\vec{\beta}$.

My strategy is the prove that the image of $\vec{\beta}$ lies in a plane. So I want to find constants a, b, c, d such that $a\beta_1 + b\beta_2 + c\beta_3 = d$. Plug in the expressions for the β_i, do a little algebra, and set the coefficients of the corresponding terms on the two sides of the equation to be equal. This gives you three linear equations in four unknowns. One solution is $a = 2$, $b = 0$, $c = -1$, $d = 1$, which produces the plane $2\beta_1 - \beta_3 = 1$. The other solutions produce constant multiples of this equation, and hence the same plane.

The intersection of this plane with the sphere S is a circle (as opposed to a single point or the empty set). Since $\vec{x}$ is a continuous bijection and $\vec{\alpha}$ is a simple closed curve, $\vec{\beta}$ must also be a simple closed curve. It's contained in the circle of intersection of the sphere and the plane. The only possibility is that the image of $\vec{\beta}$ is the entire circle.