MATH 104, SPRING 2006, ASSIGNMENT 2

JOSHUA R. DAVIS

Many students experienced difficulty on the following problems, so I have written up my solutions [with editorial comments in braces, like this].

1.2.15.

A. Let $\vec{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$. Suppose that $\vec{x} \cdot \vec{y} = 0$ for all $\vec{x} \in \mathbb{R}^n$. For $i = 1, \ldots n$, let $\vec{e_i}$ denote the ith standard basis vector in $\mathbb{R}^n$ — that is, the vector with a 1 in the ith coordinate and 0s elsewhere. (For example, $\vec{e_1} = (1, 0, 0, \ldots, 0)$ and $\vec{e_2} = (0, 1, 0, \ldots, 0)$.) Then $\vec{e_i} \cdot \vec{y} = y_i$, which equals 0 by assumption. So all the y_i are 0, so $\vec{y} = \vec{0}$.

B. Let $\vec{y}, \vec{z} \in \mathbb{R}^n$ be such that $\vec{x} \cdot \vec{y} = \vec{x} \cdot \vec{z}$ for all $\vec{x} \in \mathbb{R}^n$. Then for all $\vec{x}$,

$$0 = \vec{x} \cdot \vec{y} - \vec{x} \cdot \vec{z} = \vec{x} \cdot (\vec{y} - \vec{z}).$$

From part A we can conclude that $\vec{y} - \vec{z} = \vec{0}$, so $\vec{y} = \vec{z}$.

[Here I have explained the meaning of $\vec{e_i}$ in detail, since we have not used this notation much so far. If I were writing this later in the semester, I would not bother to explain it.]

1.3.10.

A. Let $\vec{b}$ and $\vec{c}$ be nonparallel vectors in $\mathbb{R}^3$ and $\vec{a}$ any vector in $\mathbb{R}^3$.

Assume that $\vec{a} = s\vec{b} + t\vec{c}$ for some s, t, not both 0. Let $\vec{x}$ be any vector in the intersection of the planes $\vec{b} \cdot \vec{x} = 0$ and $\vec{c} \cdot \vec{x} = 0$. Then

$$\vec{a} \cdot \vec{x} = (s\vec{b} + t\vec{c}) \cdot \vec{x} = s\vec{b} \cdot \vec{x} + t\vec{c} \cdot \vec{x} = s0 + t0 = 0.$$

So $\vec{x}$ also lies in the plane $\vec{a} \cdot \vec{x} = 0$. Therefore the intersection of the planes $\vec{b} \cdot \vec{x} = 0$ and $\vec{c} \cdot \vec{x} = 0$ is a subset of the plane $\vec{a} \cdot \vec{x} = 0$.

For the converse, assume that the intersection of the planes $\vec{b} \cdot \vec{x} = 0$ and $\vec{c} \cdot \vec{x} = 0$ is a subset of the plane $\vec{a} \cdot \vec{x} = 0$. This means that every vector $\vec{v}$ that is orthogonal to both $\vec{b}$ and $\vec{c}$ is also orthogonal to $\vec{a}$. Let P be the plane spanned by $\vec{b}$ and $\vec{c}$, and $\vec{v}$ any vector normal to this plane; then $\vec{v}$ is orthogonal to both $\vec{b}$ and $\vec{c}$, so we may conclude that it is also orthogonal to $\vec{a}$. Since $\vec{a}$ is orthogonal to $\vec{v}$, $\vec{a}$ must lie in the plane P. But P is the span of $\vec{b}$ and $\vec{c}$, so $\vec{a} = s\vec{b} + t\vec{c}$ for some s, t.

So we have proved that $\vec{a}$ is a linear combination of $\vec{b}$ and $\vec{c}$ if and only if the plane $\vec{a} \cdot \vec{x} = 0$ contains the intersection of the planes $\vec{b} \cdot \vec{x} = 0$ and $\vec{c} \cdot \vec{x} = 0$.

Geometrically, this says that a vector $\vec{a}$ lies in the plane spanned by $\vec{b}$ and $\vec{c}$ if and only if it is orthogonal to every vector $\vec{x}$ that is orthogonal to both $\vec{b}$ and $\vec{c}$.

[This is the most logically complicated problem in section 1.3, because one of the sides of the "if and only if" statement is itself an implicit "if" statement, and that "if" statement itself is complicated.]

1.3.11. Suppose that P is a plane in $\mathbb{R}^3$ spanned by the vectors $\vec{u}$ and $\vec{v}$, and that $\vec{a}$ is a vector normal to P.

A. Suppose that $\vec{u} \cdot \vec{v} = 0$. Let $\vec{x} \in P$. Then $\vec{x} = a\vec{u} + b\vec{v}$ for some a, b, and

$$\begin{aligned}
\mathrm{proj}_{\vec{u}}\vec{x} &= \frac{\vec{x} \cdot \vec{u}}{\vec{u} \cdot \vec{u}}\vec{u} \\
&= \frac{(a\vec{u} + b\vec{v}) \cdot \vec{u}}{\vec{u} \cdot \vec{u}}\vec{u} \\
&= \frac{a\vec{u} \cdot \vec{u}}{\vec{u} \cdot \vec{u}}\vec{u} + \frac{b\vec{v} \cdot \vec{u}}{\vec{u} \cdot \vec{u}}\vec{u} \\
&= a\vec{u} + 0\vec{u} \\
&= a\vec{u}.
\end{aligned}$$

A symmetric argument shows that $\mathrm{proj}_{\vec{v}}\vec{x} = b\vec{v}$. Therefore

$$\vec{x} = a\vec{u} + b\vec{v} = \mathrm{proj}_{\vec{u}}\vec{x} + \mathrm{proj}_{\vec{v}}\vec{x}.$$

B. Suppose that $\vec{u} \cdot \vec{v} = 0$. Let $\vec{x}$ be any vector in $\mathbb{R}^3$. Let $\vec{y} = \vec{x} - \mathrm{proj}_{\vec{a}}\vec{x}$, so that $\vec{x} = \mathrm{proj}_{\vec{a}}\vec{x} + \vec{y}$. Here $\mathrm{proj}_{\vec{a}}\vec{x}$, being a multiple of $\vec{a}$, is perpendicular to the plane P; I claim that the other summand, $\vec{y}$, lies in P:

$$\begin{aligned}
\vec{a} \cdot \vec{y} &= \vec{a} \cdot (\vec{x} - \mathrm{proj}_{\vec{a}}\vec{x}) \\
&= \vec{a} \cdot \vec{x} - \vec{a} \cdot \mathrm{proj}_{\vec{a}}\vec{x} \\
&= \vec{a} \cdot \vec{x} - \vec{a} \cdot \frac{\vec{x} \cdot \vec{a}}{\vec{a} \cdot \vec{a}}\vec{a} \\
&= \vec{a} \cdot \vec{x} - \frac{\vec{x} \cdot \vec{a}}{\vec{a} \cdot \vec{a}}\vec{a} \cdot \vec{a} \\
&= \vec{a} \cdot \vec{x} - \vec{x} \cdot \vec{a} \\
&= 0.
\end{aligned}$$

Therefore $\vec{a}$ and $\vec{y}$ are orthogonal and so $\vec{y}$ lies in P, as claimed. Now, since $\vec{y}$ is in P and $\vec{u} \cdot \vec{v} = 0$, part A tells us that

$$\vec{y} = \mathrm{proj}_{\vec{u}}\vec{y} + \mathrm{proj}_{\vec{v}}\vec{y}.$$

To compute each of these projections, notice first that $(\mathrm{proj}_{\vec{a}}\vec{x}) \cdot \vec{u} = 0$, since $\mathrm{proj}_{\vec{a}}\vec{x}$ is a multiple of $\vec{a}$, which is orthogonal to $\vec{u}$. Thus

$$\begin{aligned}
\mathrm{proj}_{\vec{u}}\vec{y} &= \mathrm{proj}_{\vec{u}}(\vec{x} - \mathrm{proj}_{\vec{a}}\vec{x}) \\
&= \frac{(\vec{x} - \mathrm{proj}_{\vec{a}}\vec{x}) \cdot \vec{u}}{\vec{u} \cdot \vec{u}}\vec{u} \\
&= \frac{\vec{x} \cdot \vec{u}}{\vec{u} \cdot \vec{u}}\vec{u} - \frac{(\mathrm{proj}_{\vec{a}}\vec{x}) \cdot \vec{u}}{\vec{u} \cdot \vec{u}}\vec{u} \\
&= \frac{\vec{x} \cdot \vec{u}}{\vec{u} \cdot \vec{u}}\vec{u} \\
&= \mathrm{proj}_{\vec{u}}\vec{x}.
\end{aligned}$$

A symmetric argument shows that $\mathrm{proj}_{\vec{v}}\vec{y} = \mathrm{proj}_{\vec{v}}\vec{x}$. Therefore

$$\begin{aligned}
\vec{x} &= \mathrm{proj}_{\vec{a}}\vec{x} + \vec{y} \\
&= \mathrm{proj}_{\vec{a}}\vec{x} + \mathrm{proj}_{\vec{u}}\vec{y} + \mathrm{proj}_{\vec{v}}\vec{y} \\
&= \mathrm{proj}_{\vec{a}}\vec{x} + \mathrm{proj}_{\vec{u}}\vec{x} + \mathrm{proj}_{\vec{v}}\vec{x}.
\end{aligned}$$

C. Let $\vec{u} = (1,0,0)$ and $\vec{v} = (1,1,0)$. So $\vec{u} \cdot \vec{v} = 1 \neq 0$. Let $\vec{x} = \vec{v} = (1,1,0)$, which is certainly in the plane spanned by $\vec{u}$ and $\vec{v}$. Then $\mathrm{proj}_{\vec{u}}\vec{x} = (1,0,0)$ and $\mathrm{proj}_{\vec{v}}\vec{x} = (1,1,0)$. The sum of the two projections is $(2,1,0)$, which does not equal $\vec{x}$. So the conclusion of part A need not hold if $\vec{u} \cdot \vec{v} \neq 0$.

[My written solution to part B looks nothing like the scratch work I used to figure it out! After figuring out all the pieces, I try to put them into a logical order that can be understood by the reader. The reader may not understand at first *why* I do certain steps, but she should be able to follow every step and then see how everything comes together perfectly at the end to prove the result.

In part C, I try to pick the simplest example I can find, using "simple" numbers like 0 and 1 as much as possible.]