Math 104-01, Spring 2006, Exam 2

Instructions: This is an unlimited-time, open-book take-home exam — sort of like a homework assignment on which you are not allowed to collaborate. The exam is due at the start of class on Wednesday, 29 March 2006. I anticipate that it will take longer than one day to complete — about as long as a homework assignment? — and you may find it helpful to revisit a problem over several days. So I recommend that you get started as soon as possible.

Your solutions should be polished (concise, neat, and well-written, employing complete sentences with punctuation) and self-explanatory. Submit them in a single stapled packet, presented in the order they were assigned. Always show enough work so that I can follow your solution, but do not show scratch work (false starts, circuitous reasoning, etc.). Quantitative answers should always be exact and simplified.

Partial credit is often awarded. If you cannot solve a problem, write a *brief* summary of the approaches you've tried. Exam grades will be curved, as usual.

Write and sign the honor pledge on your packet of solutions. Here are the rules:

- You may freely consult all of this class' material: the textbook, your class notes, your old homework, your old exam, and the class web site. If you missed a lecture and need to copy someone else's class notes, do so before beginning the exam.
- You may talk to me in private. You may ask clarifying questions for free. If you're really stuck on a problem, then you may ask for a hint, which will cost you some points. The opportunity to ask questions is another reason to get started early.
- You may not cite theorems from later parts of the book that we have not studied. Your solutions should make use of techniques covered thus far.
- You may not consult any other papers, books, microfiche, film, video, audio recordings, Internet sites, etc. You may not use a calculator or computer, except to view the class web site.
- You may not discuss the exam in any way (spoken, written, pantomime, etc.) with anyone but me. During the exam you will inevitably see your classmates around campus. Please refrain from asking even seemingly innocuous questions such as "Have you started the exam yet?" (If a statement or question conveys any information, then it is not allowed; if it conveys no information, then you have no reason to make it.)

If you have any questions about the exam or its rules, then ask for clarification.

1. In our discussion of 3D graphics, we studied how to rotate, translate, and project triangles in $\mathbb{R}^3$ into a plane. We did not discuss how triangles in a plane (let's say it's just $\mathbb{R}^2$) are then drawn. One problem is that in practice we don't really have an infinitely large plane on which to draw, but just a finite, usually rectangular computer screen or movie screen. For simplicity, suppose that our screen is the unit square with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$, $(0, 1)$ sitting in $\mathbb{R}^2$. A projected triangle might lie in the screen, in which case we draw it, or it might lie entirely outside the screen, in which case we don't draw it. Then again, the triangle might lie partly in the screen and partly outside it; we only want to draw the part of the triangle that lies in the screen.

Consider the figure below. The big triangle has one vertex (x_1, y_1) in the screen, and its two other vertices (x_2, y_2) and (x_3, y_3) lie outside the screen. We want to draw the intersection of the triangle with the screen, which is a pentagon. One vertex of the pentagon is (x_1, y_1), and the other four are currently unknown. To find them, we need to intersect the sides of the triangle with the sides of the screen.

A. Give a formula for the intersection point (p, q) in terms of (x_1, y_1) and (x_2, y_2).

Solution: The line through (x_1, y_1) and (x_2, y_2) has parametric form

$$\ell(t) = (x_1, y_1) + t(x_2 - x_1, y_2 - y_1) = (x_1 + t(x_2 - x_1), y_1 + t(y_2 - y_1)),$$

hitting (x_1, y_1) at time $t = 0$ and (x_2, y_2) at time $t = 1$. Similarly, the line through $(1, 0)$ and $(1, 1)$ has parametric form

$$m(s) = (1, 0) + s(0, 1) = (1, s).$$

We wish to find a single point (p, q) on both lines; that is, we wish to solve

$$\begin{aligned} x_1 + (x_2 - x_1)t &= 1 \\ y_1 + (y_2 - y_1)t &= s \end{aligned}$$

for t and s. In other words, we wish to solve $A \begin{bmatrix} t \\ s \end{bmatrix} = \begin{bmatrix} 1 - x_1 \\ -y_1 \end{bmatrix}$, where

$$A = \begin{bmatrix} x_2 - x_1 & 0 \\ y_2 - y_1 & -1 \end{bmatrix}.$$

A special case occurs when $x_1 = x_2$; then the line ℓ is vertical, so either every point between $(1, 0)$ and $(1, 1)$ is a solution or no point is. [I'll omit this special case; you should not.] In the other case, $x_2 - x_1 \neq 0$ and

$$A^{-1} = \begin{bmatrix} \frac{1}{x_2 - x_1} & 0 \\ \frac{y_2 - y_1}{x_2 - x_1} & -1 \end{bmatrix}.$$

So the solution is

$$\begin{bmatrix} t \\ s \end{bmatrix} = \begin{bmatrix} \frac{1}{x_2 - x_1} & 0 \\ \frac{y_2 - y_1}{x_2 - x_1} & -1 \end{bmatrix} \begin{bmatrix} 1 - x_1 \\ -y_1 \end{bmatrix} = \begin{bmatrix} \frac{1 - x_1}{x_2 - x_1} \\ \frac{(y_2 - y_1)(1 - x_1)}{x_2 - x_1} + y_1 \end{bmatrix}.$$

The desired point is

$$\begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} 1 \\ s \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{(y_2 - y_1)(1 - x_1)}{x_2 - x_1} + y_1 \end{bmatrix}.$$

B. In general, if I give you any two points (x_1, y_1) and (x_2, y_2) in $\mathbb{R}^2$ (not necessarily arranged as in the figure below — they can now be anywhere), then the line segment between the two might not intersect the screen at all, in which case the computation from Part A is not really useful. Given any two points (x_1, y_1) and (x_2, y_2), find a specific criterion that detects whether the line segment between them intersects the line segment between $(1, 0)$ and $(1, 1)$.

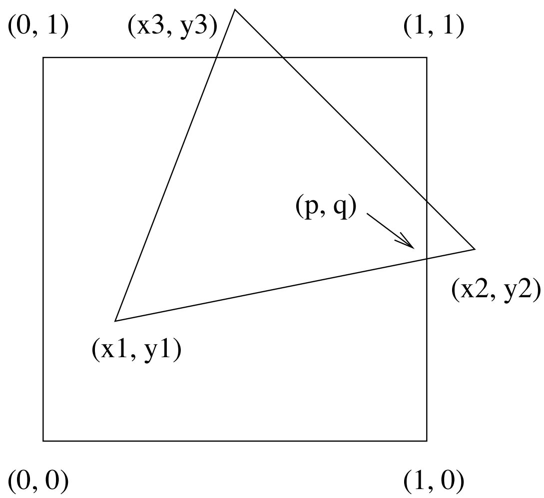
Solution: In the notation of Part A, the line segments in question are $\ell(t)$ for $0 \leq t \leq 1$ and $m(s)$ for $0 \leq s \leq 1$. So we simply compute t and s using the formulas of Part A; the line segments intersect if and only if both

$$t = \frac{1 - x_1}{x_2 - x_1}$$

and

$$s = \frac{(y_2 - y_1)(1 - x_1)}{x_2 - x_1} + y_1$$

are between 0 and 1.



2. For any vectors $\vec{v} = (v_1, v_2)$ and $\vec{w} = (w_1, w_2)$ in $\mathbb{R}^2$, define

$$\langle \vec{v}, \vec{w} \rangle = v_1 w_1 - v_2 w_2.$$

This is similar to the dot product, but with a minus sign. Define $||\vec{v}|| = \sqrt{\langle \vec{v}, \vec{v} \rangle}$, as one usually defines a norm from an inner product.

A. It turns out that this $\langle \vec{v}, \vec{w} \rangle$ does not define an inner product on $\mathbb{R}^2$. Which parts of the definition of inner product does it satisfy, and which parts does it not satisfy?

Solution: [It satisfies the addition, scalar multiplication, and symmetry properties, as you should show; I'll omit these.] It does not satisfy positive definiteness; for example, if $\vec{v} = (1, 1)$, then $\langle \vec{v}, \vec{v} \rangle = 1 - 1 = 0$, even though $\vec{v} \neq \vec{0}$.

B. It also turns out that $||\vec{v}||$ doesn't define a norm; for one thing, it's not even defined everywhere. For which vectors $\vec{v} \in \mathbb{R}^2$ is $||\vec{v}||$ defined? For which $\vec{v}$ is it 0? For which $\vec{v}$ is it 1? Answer these questions both in words/equations and in a detailed sketch of $\mathbb{R}^2$.

Solution: The quantity $||\vec{v}|| = ||(v_1, v_2)|| = \sqrt{v_1^2 - v_2^2}$ is 0 for $\vec{v}$ such that $v_1 = \pm v_2$; in the v_1-v_2-plane, this is the "cone" consisting of the two lines of slope ± 1 through the origin. This cone divides the plane into four regions: left, right, top, and bottom. The quantity $||\vec{v}||$ is undefined when $v_1^2 < v_2^2$; this occurs in the top and bottom regions. In the other regions it is defined. The quantity $||\vec{v}||$ equals 1 if and only if $v_1^2 - v_2^2 = 1$, which describes a hyperbola. [I'll omit the sketch; you should not.]

C. Terminology: The *determinant* of a 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is defined to be the quantity $ad - bc$. A matrix is said to be *special* if it has determinant 1.

In Exercise 2.3 #13 you showed that all 2×2 special orthogonal matrices are of the form

$$A_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

for some number θ. (The other form in the exercise has determinant -1, as you can check; I'm not interested in that.) In Exercise 2.1 #9a you also proved that these matrices preserve the standard norm on $\mathbb{R}^2$, meaning that the norm of $A_\theta \vec{v}$ equals the norm of $\vec{v}$.

Now I want you to do the same thing for the fake norm $||\vec{v}||$ defined above. That is, give a formula for matrices B_θ such that the determinant of B_θ is 1 and $||B_\theta \vec{v}||^2 = ||\vec{v}||^2$ for all $\vec{v} \in \mathbb{R}^2$. Explicitly show that your B_θ satisfies both of these properties. (Hint: Instead of using cos and sin, use cosh and sinh. Recall that these are defined as $\cosh\theta = (e^\theta + e^{-\theta})/2$ and $\sinh\theta = (e^\theta - e^{-\theta})/2$. They satisfy the identities such as $\cosh^2\theta - \sinh^2\theta = 1$, $\cosh(-\theta) - \cosh\theta$, and $\sinh(-\theta) = -\sinh\theta$. Be careful about sign errors.)

Solution: I propose the answer

$$B_\theta = \begin{bmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{bmatrix}.$$

This has determinant $\cosh^2\theta - \sinh^2\theta = 1$. Also,

$$\begin{aligned}
||B_\theta \vec{v}||^2 &= \left|\left|\begin{bmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right|\right|^2 \\
&= \left|\left|\begin{bmatrix} v_1\cosh\theta + v_2\sinh\theta \\ v_1\sinh\theta + v_2\cosh\theta \end{bmatrix}\right|\right|^2 \\
&= (v_1\cosh\theta + v_2\sinh\theta)^2 - (v_1\sinh\theta + v_2\cosh\theta)^2 \\
&= v_1^2\cosh^2\theta + v_1v_2\cosh\theta\sinh\theta + v_2^2\sinh^2\theta - v_1^2\sinh^2\theta - v_1v_2\sinh\theta\cosh\theta - v_2^2\cosh^2\theta \\
&= v_1^2(\cosh^2\theta - \sinh^2\theta) - v_2^2(\cosh^2\theta - \sinh^2\theta) \\
&= v_1^2 - v_2^2 \\
&= ||\vec{v}||^2.
\end{aligned}$$

[Remark: Under the standard inner product on $\mathbb{R}^2$, the vectors of length 1 constitute a circle, and we use circular trig functions to describe rotations. Under the fake inner product of this problem, the vectors of length 1 constitute a hyperbola, and we use hyperbolic trig functions to describe "rotations". Coincidence? No.]

[Remark: The fake inner product used in this exercise is essentially two-dimensional special relativity, with v_1 measuring one dimension of space and v_2 measuring time. The cone described in Part B is called the *light cone*; a particle moving at light speed travels along this cone.]

3. Let A be any $n \times n$ skew-symmetric matrix (which means that $A^\top = -A$, recall).

A. Prove that the kth power of A is symmetric if k is even and skew-symmetric if k is odd.

Solution: We compute

$$(A^k)^\top = (\underbrace{AA\cdots A}_{k\text{ times}})^\top = \underbrace{A^\top A^\top \cdots A^\top}_{k\text{ times}} = \underbrace{(-A)(-A)\cdots(-A)}_{k\text{ times}} = (-1)^k \underbrace{AA\cdots A}_{k\text{ times}} = (-1)^k A^k.$$

So $(A^k)^\top = A^k$ if k is even and $(A^k)^\top = -A^k$ if k is odd, as desired.

B. For any odd number k, prove that $\vec{v} \cdot (A^k \vec{v}) = 0$ for all $\vec{v} \in \mathbb{R}^n$.

Solution: Let k be odd; then Part A says that $(A^k)^\top = -A^k$. For any $\vec{v} \in \mathbb{R}^n$,

$$\begin{aligned}
\vec{v} \cdot (A^k \vec{v}) &= \vec{v}^\top A^k \vec{v} \\
&= \left(\vec{v}^\top A^k \vec{v}\right)^\top \\
&= \vec{v}^\top (A^k)^\top (\vec{v}^\top)^\top \\
&= \vec{v}^\top (-A^k) \vec{v} \\
&= -\left(\vec{v}^\top A^k \vec{v}\right) \\
&= -\left(\vec{v} \cdot (A^k \vec{v})\right).
\end{aligned}$$

(The second equality holds because any 1×1 matrix equals its transpose.) So $\vec{v} \cdot (A^k \vec{v})$ equals its own negation; it must be 0.

C. Prove that if $n = 3$ then A must be singular. (In fact, A must be singular for any odd n, but you don't have to prove that.)

Solution: A 3×3 skew-symmetric matrix A must be of the form

$$A = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}.$$

If $a = 0$, then the first and second row vectors are parallel, so A is singular. If $a \neq 0$, then we perform Gaussian elimination as follows. [I'll just describe it; you should show the steps, with no explanation necessary. First switch the first and second rows. Then use these to zero out the first two entries of the third row; the third entry also gets zeroed. The resulting matrix is in echelon form, with the last row all zero.] Therefore A is singular.

[Remark: Nonsingular skew-symmetric matrices are sometimes called *symplectic*; they lie at the heart of symplectic topology, a kind of math that is used both in classical mechanics and in current theoretical physics.]

4. In Section 3.6 (say, Example 1d) we saw several examples of vector spaces of functions. Now we will generalize all of them. Let X be any set and V any vector space (not necessarily finite-dimensional). Let F be the set of all functions from X to V. (All you know about a function $f \in F$ is that for each element $x \in X$ it produces a vector $f(x) \in V$.)

A. Show that F is a vector space under the operations of function addition and scalar multiplication. (We say that "F inherits a vector space structure from V".)

Solution: [You should check the eight vector space axioms on page 199. This is identical to verifying that any other space of functions is a vector space, as you have done before on your homework.]

B. Let x be any particular element of X and $F_x \subseteq F$ the set of functions f such that $f(x) = \vec{0}$. Show that F_x is a subspace of F.

Solution: The zero function $0 : X \to V$, which has constant value $\vec{0}$, certainly has value $\vec{0}$ at x, so F_x does contain the zero element of F. If f and g are in F_x, then $f(x) = 0 = g(x)$, so $(f + g)(x) = f(x) + g(x) = \vec{0} + \vec{0} = \vec{0}$. So F_x is closed under addition. If f is in F_x, then for any $c \in \mathbb{R}$ we have $(cf)(x) = cf(x) = c\vec{0} = \vec{0}$, so cf is in F_x. Thus F_x is closed under scalar multiplication. So F_x is a subspace of F.

C. Suppose that V possesses an inner product $\langle\, , \rangle$. Suppose also that the set X is finite; say $X = \{x_1, \ldots, x_k\}$. Define

$$\langle\!\langle f, g \rangle\!\rangle = \sum_{i=1}^{k} \langle f(x_i), g(x_i) \rangle.$$

Is this $\langle\!\langle\, , \rangle\!\rangle$ an inner product on F? Prove or disprove it.

Solution: It is an inner product; each property is inherited from the inner product $\langle\, , \rangle$ on V, as follows. For addition,

$$\begin{aligned}
\langle\!\langle f_1 + f_2, g \rangle\!\rangle &= \sum_{i=1}^{k} \langle (f_1 + f_2)(x_i), g(x_i) \rangle \\
&= \sum_{i=1}^{k} \langle f_1(x_i) + f_2(x_i), g(x_i) \rangle \\
&= \sum_{i=1}^{k} \langle f_1(x_i), g(x_i) \rangle + \langle f_2(x_i), g(x_i) \rangle \\
&= \sum_{i=1}^{k} \langle f_1(x_i), g(x_i) \rangle + \sum_{i=1}^{k} \langle f_2(x_i), g(x_i) \rangle \\
&= \langle\!\langle f_1, g \rangle\!\rangle + \langle\!\langle f_2, g \rangle\!\rangle.
\end{aligned}$$

For scalar multiplication,

$$\langle\!\langle cf, g \rangle\!\rangle = \sum_{i=1}^{k} \langle (cf)(x_i), g(x_i) \rangle$$

$$
\begin{aligned}
&= \sum_{i=1}^{k} \langle c f(x_i), g(x_i) \rangle \\
&= \sum_{i=1}^{k} c \langle f(x_i), g(x_i) \rangle \\
&= c \sum_{i=1}^{k} \langle f(x_i), g(x_i) \rangle \\
&= c \langle\!\langle f, g \rangle\!\rangle .
\end{aligned}
$$

For symmetry,

$$
\begin{aligned}
\langle\!\langle f, g \rangle\!\rangle &= \sum_{i=1}^{k} \langle f(x_i), g(x_i) \rangle \\
&= \sum_{i=1}^{k} \langle g(x_i), f(x_i) \rangle \\
&= \langle\!\langle g, f \rangle\!\rangle .
\end{aligned}
$$

For positive-definiteness, assume that f is a nonzero function. This means that $f(x) \neq \vec{0}$ for some $x \in X$. Then the positive-definiteness of $\langle\ ,\ \rangle$ implies that $\langle f(x), f(x) \rangle > 0$ for this x, and also that $\langle f(x_i), f(x_i) \rangle \geq 0$ for the other x_i. So $\langle\!\langle f, f \rangle\!\rangle = \sum_{i=1}^{k} \langle f(x_i), f(x_i) \rangle > 0$.

D. Again suppose that $X = \{x_1, \ldots, x_k\}$ is a finite set. Also suppose that V is of finite dimension n. Determine the dimension of F, and find a basis for it. (Suggestion: You might want to try a few small examples first.)

Solution: Let $\{\vec{v_1}, \ldots, \vec{v_n}\}$ be a basis for V. For each $i = 1, \ldots, k$ and $j = 1, \ldots, n$, define a function $f_{ij} : X \to V$ by

$$
\begin{aligned}
f_{ij}(x) = \vec{v_j} &\quad \text{if} \quad x = x_i, \\
f_{ij}(x) = \vec{0} &\quad \text{for} \quad x \neq x_i.
\end{aligned}
$$

This set of kn functions forms a basis for F; thus F is kn-dimensional. [You should check that this set spans and is independent.]

5. In $\mathbb{R}^4$, let $\vec{u} = \begin{bmatrix} 1 \\ -2 \\ 2 \\ 0 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$, and $\vec{w} = \begin{bmatrix} -1 \\ -2 \\ 4 \\ 2 \end{bmatrix}$. Find a basis for the orthogonal complement of the span of $\{\vec{u}, \vec{v}, \vec{w}\}$.

Solution: Let A be the 3×4 matrix with rows $\vec{u}$, $\vec{v}$, $\vec{w}$. We wish to find the orthogonal complement of the row space of A, which is the null space of A; that is, we wish to solve $A\vec{x} = \vec{0}$. [I'll omit the work here; you should not.] A basis for the null space is $\{\langle 2, 3, 2, 0 \rangle, \langle 2, 1, 0, 2 \rangle\}$.

6. You may already know the formulae

$$1+2+\cdots+n=\frac{1}{2}n^2+\frac{1}{2}n, \qquad 1^2+2^2+\cdots+n^2=\frac{1}{3}n^3+\frac{1}{2}n^2+\frac{1}{6}n.$$

I want to find a similar formula for the sum of fourth powers. Fortunately, there is a theorem that says that, for any positive integer k (in my case, $k=4$) there exist constants a_1, a_2, $\ldots$, a_{k+1} such that for all $n \geq 1$,

$$\sum_{i=1}^{n} i^k = a_{k+1}n^{k+1} + a_k n^k + \cdots + a_1 n.$$

You do not have to prove this theorem, but do explain how you would use linear algebra to find the promised formula for $k=4$. That is, write a system of linear equations whose solution gives the coefficients a_i in the $k=4$ formula. (You do not have to solve the linear system.)

Solution: We have five unknown quantities, so we want five independent equations. The formula must hold for all n, so in particular it must hold for $n=1,2,3,4,5$. Plugging in these values of n, we see that the a_i must satisfy

$$\begin{aligned}
a_5 1^5 + a_4 1^4 + a_3 1^3 + a_2 1^2 + a_1 1^1 &= 1 \\
a_5 2^5 + a_4 2^4 + a_3 2^3 + a_2 2^2 + a_1 2^1 &= 17 \\
a_5 3^5 + a_4 3^4 + a_3 3^3 + a_2 3^2 + a_1 3^1 &= 98 \\
a_5 4^5 + a_4 4^4 + a_3 4^3 + a_2 4^2 + a_1 4^1 &= 354 \\
a_5 5^5 + a_4 5^4 + a_3 5^3 + a_2 5^2 + a_1 5^1 &= 979.
\end{aligned}$$

[By the way, the solution is $\langle a_5, a_4, a_3, a_2, a_1\rangle = \langle 1/5, 1/2, 1/3, 0, -1/30\rangle$, so

$$\sum_{i=1}^{n} i^4 = \frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n.$$

This was not required.]