Math 104-01, Spring 2006, Exam 1 Answers

Instructions: There are five pages of problems. Each is worth 20%. You have 50 minutes. Calculators are not allowed.

Always **show all of your work**. Pictures are often helpful. Partial credit will be awarded. Give **simplified**, **exact** answers, and make sure they are clearly marked.

If you are unsure about how you should interpret any question, ask me. If you are unable to begin a problem because you don't remember a key formula or can't figure out how to set up the problem, you may ask me for a hint. The hint will cost you some points.

1. In $\mathbb{R}^n$ I have k vectors $\vec{v}_1, \ldots, \vec{v}_k$. Their span is some subset of $\mathbb{R}^n$, and I want to understand it. To be specific, my problem is to find a system of linear equations whose solution set is the span in question.

A. Describe in words how you would solve this problem, in general.

Solution: Construct an $n \times k$ matrix A with the vectors $\vec{v}_i$ as its columns. Augment this matrix with the vector $\vec{b} = \langle b_1, \ldots, b_n \rangle$. Then do Gaussian elimination to bring $A|\vec{b}$ into echelon form $U|\vec{c}$. Each row of zeros in U gives a constraint equation. (The corresponding entry of $\vec{c}$ must equal 0. These constraint equations form the system I wanted.

B. Explicitly solve the problem for the particular case when $n = 4$, $k = 3$, $\vec{v}_1 = \langle 1, 0, -1, 2 \rangle$, $\vec{v}_2 = \langle 0, 3, 1, 0 \rangle$, and $\vec{v}_3 = \langle 1, 6, 5, 3 \rangle$.

Solution: The span is the hyperplane described by the equation $(-9/4)b_1 + (1/12)b_2 + (-1/4)b_3 + b_4 = 0$. I'll leave the work to the reader.

2. In $\mathbb{R}^n$ I have k hyperplanes through the origin with normals $\vec{a}_1, \ldots, \vec{a}_k$. The intersection of these hyperplanes is some subset of $\mathbb{R}^n$, and I want to understand it. To be specific, my problem is to find a parametrization of the intersection.

A. Describe in words how you would solve this problem, in general.

Solution: Construct a $k \times n$ matrix A with the vectors $\vec{a}_i$ as its rows. Then solve the matrix equation $A\vec{x} = \vec{0}$ in the usual way, by reducing A (or, equivalently, $A|\vec{0}$) to reduced echelon form, letting the variables x_i without a leading 1 in their columns be free, and expressing the other variables in terms of these free variables.

B. Explicitly solve the problem for the particular case when $n = 4$, $k = 2$, $\vec{a}_1 = \langle 0, 0, 2, 4 \rangle$, and $\vec{a}_2 = \langle 1, 3, -1, 0 \rangle$.

Solution: I find x_4 and x_2 to be free, with $x_3 = -2x_4$ and $x_1 = -3x_2 - 2x_4$. So a parametrization is $\vec{x} = x_2 \langle -3, 1, 0, 0 \rangle + x_4 \langle -2, 0, -2, 1 \rangle$. I'll leave the work to the reader.

3. [Preamble: The projection of $\vec{v}$ onto $\vec{w}$ is denoted $\text{proj}_{\vec{w}} \vec{v}$ or sometimes $\vec{v}^{||}$. Once this is defined, one can define $\vec{v}^{\perp} = \vec{v} - \vec{v}^{||}$, so that $\vec{v} = \vec{v}^{||} + \vec{v}^{\perp}$ with $\vec{v}^{||}$ parallel to the line $t\vec{w}$ and $\vec{v}^{\perp}$ orthogonal to this line. In your homework (2.1#7) you have described such projections in $\mathbb{R}^2$ using 2×2 matrices. In this problem we carry out the same computation, but in $\mathbb{R}^3$, and projecting onto a plane rather than a line.]

In $\mathbb{R}^3$ let $\vec{a} = \langle a_1, a_2, a_3 \rangle$ be any nonzero vector and P the plane through the origin with normal

$\vec{a}$. For any vector $\vec{v} = \langle v_1, v_2, v_3 \rangle$, let

$$\vec{v}^{\perp} = \mathrm{proj}_{\vec{a}}\vec{v} \qquad \text{and} \qquad \vec{v}^{||} = \vec{v} - \vec{v}^{\perp},$$

so that $\vec{v} = \vec{v}^{||} + \vec{v}^{\perp}$. Since $\vec{v}^{\perp}$ is a multiple of $\vec{a}$, it must be normal to the plane P.

A. Prove that $\vec{v}^{||}$ lies in the plane P.

Solution: We compute

$$\begin{aligned}
\vec{a} \cdot \vec{v}^{||} &= \vec{a} \cdot (\vec{v} - \mathrm{proj}_{\vec{a}}\vec{v}) \\
&= \vec{a} \cdot \vec{v} - \vec{a} \cdot \frac{\vec{v} \cdot \vec{a}}{\vec{a} \cdot \vec{a}}\vec{a} \\
&= \vec{a} \cdot \vec{v} - \frac{\vec{v} \cdot \vec{a}}{\vec{a} \cdot \vec{a}}\vec{a} \cdot \vec{a} \\
&= \vec{a} \cdot \vec{v} - \vec{v} \cdot \vec{a} \\
&= 0.
\end{aligned}$$

Therefore $\vec{v}^{||}$ is orthogonal to $\vec{a}$, and so it lies in P.

B. Find a 3×3 matrix A that represents projection onto the plane P. That is, find A such that $A\vec{v} = \vec{v}^{||}$ for all vectors $\vec{v} \in \mathbb{R}^3$. (A will depend on $\vec{a}$, of course.)

Solution:

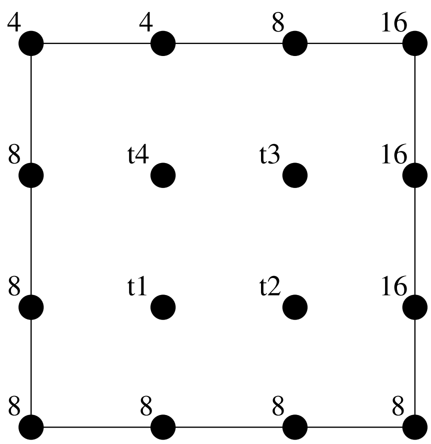
$$\begin{aligned}
\vec{v}^{||} = \vec{v} - \mathrm{proj}_{\vec{a}}\vec{v} &= \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} - \frac{\vec{v} \cdot \vec{a}}{\vec{a} \cdot \vec{a}} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \\
&= \begin{bmatrix} v_1 - \frac{a_1}{\vec{a}\cdot\vec{a}}(a_1v_1 + a_2v_2 + a_3v_3) \\ v_2 - \frac{a_2}{\vec{a}\cdot\vec{a}}(a_1v_1 + a_2v_2 + a_3v_3) \\ v_3 - \frac{a_3}{\vec{a}\cdot\vec{a}}(a_1v_1 + a_2v_2 + a_3v_3) \end{bmatrix} \\
&= \begin{bmatrix} \left(1 - \frac{a_1a_1}{\vec{a}\cdot\vec{a}}\right) v_1 - \frac{a_1a_2}{\vec{a}\cdot\vec{a}}v_2 - \frac{a_1a_3}{\vec{a}\cdot\vec{a}}v_3 \\ -\frac{a_1a_2}{\vec{a}\cdot\vec{a}}v_1 + \left(1 - \frac{a_2a_2}{\vec{a}\cdot\vec{a}}\right) v_2 - \frac{a_2a_3}{\vec{a}\cdot\vec{a}}v_3 \\ -\frac{a_1a_3}{\vec{a}\cdot\vec{a}}v_1 - \frac{a_2a_3}{\vec{a}\cdot\vec{a}}v_2 + \left(1 - \frac{a_3a_3}{\vec{a}\cdot\vec{a}}\right) v_3 \end{bmatrix} \\
&= \begin{bmatrix} \left(1 - \frac{a_1a_1}{\vec{a}\cdot\vec{a}}\right) & -\frac{a_1a_2}{\vec{a}\cdot\vec{a}} & -\frac{a_1a_3}{\vec{a}\cdot\vec{a}} \\ -\frac{a_1a_2}{\vec{a}\cdot\vec{a}} & \left(1 - \frac{a_2a_2}{\vec{a}\cdot\vec{a}}\right) & -\frac{a_2a_3}{\vec{a}\cdot\vec{a}} \\ -\frac{a_1a_3}{\vec{a}\cdot\vec{a}} & -\frac{a_2a_3}{\vec{a}\cdot\vec{a}} & \left(1 - \frac{a_3a_3}{\vec{a}\cdot\vec{a}}\right) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}.
\end{aligned}$$

[By the way, the matrix here is $\mathbb{I} - \frac{\vec{a}\vec{a}^T}{\vec{a}^T\vec{a}}$, where $\mathbb{I}$ is the identity, $\vec{a}^T\vec{a}$ is the dot product (also called the "inner" product), and $\vec{a}\vec{a}^T$ is called the "outer" product. In other words, the matrix that performs the projection $\mathrm{proj}_{\vec{a}}$ is the outer product of $\vec{a}$ with itself, divided by the inner product of $\vec{a}$ with itself.]

4. A materials lab in suburban Madagascar is studying low-temperature conductivity in metal. The technicians have set up a thin metal plate measuring 3 cm $\times$ 3 cm. While their experiment is running they are able to measure the temperatures along the boundary of the plate but not in the interior of it. They need to be able to infer the interior temperatures from the boundary temperatures. This is rather difficult, so they make the following approximation.

The technicians place an imaginary 4×4 grid of points over the plate and measure the temperatures at the twelve boundary grid points as in the figure. The temperatures t_1, t_2, t_3, t_4 in the interior are unknown, but it is reasonable to assume that *at each interior grid point, the temperature is the average of the temperatures at the eight grid points around it.*

Using this assumption, find t_1, t_2, t_3, and t_4. [Make it very clear which equations you are solving, which matrix you are reducing, etc. This earns most of the points. Only solve for the t_i if you have time. If you need extra space, use page 1.]



Solution: We have equations

$$\begin{aligned} 8t_1 &= t_2 + t_3 + t_4 + 40 \\ 8t_2 &= t_1 + t_3 + t_4 + 56 \\ 8t_3 &= t_1 + t_2 + t_4 + 60 \\ 8t_4 &= t_1 + t_2 + t_3 + 32. \end{aligned}$$

So we want to solve

$$\begin{bmatrix} 8 & -1 & -1 & -1 \\ -1 & 8 & -1 & -1 \\ -1 & -1 & 8 & -1 \\ -1 & -1 & -1 & 8 \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{bmatrix} = \begin{bmatrix} 40 \\ 56 \\ 60 \\ 32 \end{bmatrix}.$$

It turns out that the solution is $t_1 = 388/45$, $t_2 = 52/5$, $t_3 = 488/45$, $t_4 = 116/15$.

5. Let A be any $n \times n$ matrix, $\mathbb{I}$ the $n \times n$ identity matrix, and $\mathbb{O}$ the $n \times n$ zero matrix. In this problem, show in detail any matrix algebra you do; do not skip steps.

A. Assume that $A^2 = \mathbb{O}$. Find an inverse for the matrix $\mathbb{I} - A$.

Solution: Playing around a bit, we find

$$(\mathbb{I} - A)(\mathbb{I} + A) = (\mathbb{I} - A)\mathbb{I} + (\mathbb{I} - A)A = \mathbb{I}^2 - A\mathbb{I} + \mathbb{I}A - A^2 = \mathbb{I} - A + A - \mathbb{O} = \mathbb{I}.$$

Similarly $(\mathbb{I} + A)(\mathbb{I} - A) = \mathbb{I}$. So the inverse of $\mathbb{I} - A$ is $\mathbb{I} + A$.

B. Assume that $A^k = \mathbb{O}$ for some $k \geq 2$. Find an inverse for $\mathbb{I} - A$. [You have already done the $k = 2$ case; you might try $k = 3$ next, to help you see the pattern.]

Solution: By similar reasoning, the inverse of $\mathbb{I} - A$ is $\mathbb{I} + A + A^2 + \cdots + A^{k-1}$, as you can check.

[By the way, this is essentially 2.2 #19. Also, anyone who knows their power series knows that $1/(1 - x) = 1 + x + x^2 + x^3 + \cdots$ wherever the series converges. If some power of x is zero, then the series truncates and thus automatically converges. The only real number that has 0 as a power is 0, but many matrices have $\mathbb{O}$ as a power, and for these matrices the truncated power series corroborates our answers above.]

6. Let A be an $m \times n$ matrix and $\vec{b}$ a vector in $\mathbb{R}^m$. Assume that $A\vec{x} = \vec{b}$ is consistent. What relationships (equalities or inequalities) exist among the following four numbers: m, n, the rank r of A, and the number p of free parameters used in describing the solutions of $A\vec{x} = \vec{b}$?

Solution: $r \leq m$, $r \leq n$, $p = n - r$.