Vector algebra^{TW}

$$\vec{\mathbf{b}} = \langle b_1, b_2, b_3\rangle \qquad \vec{\mathbf{c}} = \langle c_1, c_2, c_3\rangle$$

$$\vec{\mathbf{a}} = \langle a_1, a_2, a_3\rangle = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}} = |\vec{\mathbf{a}}|\hat{\mathbf{a}}$$

$$k\vec{\mathbf{a}} = \langle ka_1, ka_2, ka_3\rangle$$

length $\quad |\vec{\mathbf{a}}| = \sqrt{a_1^2 + a_2^2 + a_3^2} = \sqrt{\vec{\mathbf{a}}\cdot\vec{\mathbf{a}}} \ge 0$

direction: unit vector $\quad \hat{\mathbf{a}} = \frac{1}{|\vec{\mathbf{a}}|}\vec{\mathbf{a}} \qquad |\hat{\mathbf{a}}| = 1$

$$\vec{\mathbf{a}}\cdot\vec{\mathbf{b}} = a_1b_1 + a_2b_2 + a_3b_3$$

$$\begin{array}{c} \\ \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \\ \\ \end{array} \begin{array}{ccccc} & & - & - & - \\ a_1 & a_2 & a_3 & a_1 & b_1 \\ b_1 & b_2 & b_3 & a_2 & b_2 \\ c_1 & c_2 & c_3 & a_3 & b_3 \\ & & + & + & + \end{array}$$

$$\vec{\mathbf{a}}\times\vec{\mathbf{b}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \qquad \vec{\mathbf{a}}\cdot(\vec{\mathbf{b}}\times\vec{\mathbf{c}}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Vector geometry

"Tip to Tail" graphical vector addition
"Edge vectors" connect vertex positions

$$\vec{\mathbf{a}} = \vec{PQ} = \vec{Q} - \vec{P}$$

Dot product $\quad \vec{\mathbf{a}}\cdot\vec{\mathbf{b}}$

$$\vec{\mathbf{a}}\cdot\vec{\mathbf{b}} = |\vec{\mathbf{a}}||\vec{\mathbf{b}}|\cos\theta$$

$$\vec{\mathbf{a}}\cdot\vec{\mathbf{b}} = 0 \quad\leftrightarrow\quad \vec{\mathbf{a}}\perp\vec{\mathbf{b}} \qquad \theta = 90°$$

Parallel vectors $\quad \vec{\mathbf{a}}\,||\,\vec{\mathbf{b}}$

$$\vec{\mathbf{a}}\,||\,\vec{\mathbf{b}} \quad\leftrightarrow\quad \vec{\mathbf{a}} = k\vec{\mathbf{b}}$$

$$\vec{\mathbf{a}}\,||\,\vec{\mathbf{b}} \quad\leftrightarrow\quad \vec{\mathbf{a}}\times\vec{\mathbf{b}} = \vec{\mathbf{0}}$$

Cross product $\quad \vec{\mathbf{a}}\times\vec{\mathbf{b}}$

$$(\vec{\mathbf{a}}\times\vec{\mathbf{b}})\perp\vec{\mathbf{a}} \qquad (\vec{\mathbf{a}}\times\vec{\mathbf{b}})\perp\vec{\mathbf{b}}$$

"Right-hand rule" for $\vec{\mathbf{c}} = \vec{\mathbf{a}}\times\vec{\mathbf{b}}$

$$|\vec{\mathbf{a}}\times\vec{\mathbf{b}}| = |\vec{\mathbf{a}}||\vec{\mathbf{b}}|\sin\theta \qquad 0° \le \theta \le 180°$$

$$\vec{\mathbf{a}}\,||\,\vec{\mathbf{b}} \quad\leftrightarrow\quad \vec{\mathbf{a}}\times\vec{\mathbf{b}} = \vec{\mathbf{0}} \qquad \theta = 0°, 180°$$

$|\vec{\mathbf{a}}\times\vec{\mathbf{b}}|$ = area of parallelogram

$\frac{1}{2}|\vec{\mathbf{a}}\times\vec{\mathbf{b}}|$ = area of triangle

$|\vec{\mathbf{c}}\cdot(\vec{\mathbf{a}}\times\vec{\mathbf{b}})|$ = volume of brick

Orthogonal decomposition: $||, \perp$ **projections**

Components of $\vec{\mathbf{a}}$ parallel and perpendicular to $\vec{\mathbf{b}}$

$$\vec{\mathbf{a}} = \vec{\mathbf{a}}_{||} + \vec{\mathbf{a}}_{\perp}$$

$\vec{\mathbf{a}}_{||}$ = parallel or **tangent**

$\vec{\mathbf{a}}_{\perp}$ = perpendicular or **normal**

$$\vec{\mathbf{a}}_{||} = \frac{\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}}{|\vec{\mathbf{b}}|^2}\vec{\mathbf{b}} \qquad \text{then} \qquad \vec{\mathbf{a}}_{\perp} = \vec{\mathbf{a}} - \vec{\mathbf{a}}_{||}$$

Distance problems: $\perp$ minimum distance can only be found between two things[1] that are "parallel,"[2] they must have a common normal vector $\vec{\mathbf{n}}$. Then the $\perp$ distance is given by the length of $\vec{\mathbf{v}}_{||}$, where $\vec{\mathbf{v}} = \vec{PQ} = \vec{Q} - \vec{P}$, is the vector between any points on the two "things"

$$\text{distance} = |\vec{\mathbf{v}}_{||}| = \frac{|\vec{\mathbf{v}}\cdot\vec{\mathbf{n}}|}{|\vec{\mathbf{n}}|}$$

Planes– geometry: $\vec{\mathbf{n}}\perp(\vec{\mathbf{r}} - \vec{\mathbf{r}}_0)$
normal $\vec{\mathbf{n}} = \langle a, b, c\rangle$, points in plane, $\vec{\mathbf{r}} = \langle x, y, x\rangle$

$$\vec{\mathbf{n}}\cdot\vec{\mathbf{r}} = d \qquad ax + by + cz = d$$

Lines– geometry: $\ell\,||\,\vec{\mathbf{v}}$ thru pt $\vec{\mathbf{r}}_0$
velocity vector $\vec{\mathbf{v}} = \langle a, b, c\rangle$

$$\vec{\mathbf{r}} = \vec{\mathbf{r}}_0 + t\vec{\mathbf{v}}$$

Parametric curves and Vector functions

Position

$$\vec{\mathbf{r}}(t) = \langle x(t), y(t), z(t)\rangle$$

Velocity

$$\vec{\mathbf{v}}(t) = \frac{d\vec{\mathbf{r}}}{dt} = \langle x'(t), y'(t), z'(t)\rangle$$

Acceleration

$$\vec{\mathbf{a}}(t) = \frac{d^2\vec{\mathbf{r}}}{dt^2} = \langle x''(t), y''(t), z''(t)\rangle$$

Orthogonal decomposition of $\vec{\mathbf{a}}$: $\vec{\mathbf{T}}\cdot\vec{\mathbf{N}} = 0$

$$\vec{\mathbf{a}} = a_T\vec{\mathbf{T}} + a_N\vec{\mathbf{N}}$$

unit **Tangent** vector

$$\vec{\mathbf{T}} = \hat{\mathbf{v}} = \frac{1}{|\vec{\mathbf{v}}|}\vec{\mathbf{v}}$$

unit **Normal** vector

$$\vec{\mathbf{N}} = \frac{1}{\left|\frac{d\vec{\mathbf{T}}}{dt}\right|}\frac{d\vec{\mathbf{T}}}{dt}$$

Tangential acceleration (speed-up)

$$a_T = \frac{d|\vec{\mathbf{v}}|}{dt} = \vec{\mathbf{a}}\cdot\vec{\mathbf{T}} = \frac{\vec{\mathbf{a}}\cdot\vec{\mathbf{v}}}{|\vec{\mathbf{v}}|}$$

Normal acceleration (change of direction)

$$a_N = \kappa|\vec{\mathbf{v}}|^2 \qquad a_N\vec{\mathbf{N}} = \vec{\mathbf{a}} - a_T\vec{\mathbf{T}}$$

Curvature

$$\kappa = \frac{1}{|\vec{\mathbf{v}}|}\left|\frac{d\vec{\mathbf{T}}}{dt}\right|$$

[1] things= points, lines or planes
[2] parallel or skew, they don't touch