Multiple integrals^{TW}

Integrand function $f(x,y,z)$, domain of integration D, $\int\int_D f\,dD$ or $\int\int\int_D f\,dD$.

1) Pick coordinate system: xyz (rectangular), $r\theta$ (polar, cylindrical or spherical), or uvw (general change of variables, Jacobian) $\to dD$.

$$dA = d\text{Area} \qquad dV = d\text{Volume} \qquad dS = d\text{SurfaceArea} \qquad ds = d\text{Arclength}$$

2) Write boundaries of domain of integration: limits of integration

3) Pick order of integration, do iterated integrals one variables at a time.

Rect	Cyl	Sph	General
$x = x$	$x = r\cos\theta$	$x = \rho\sin\phi\cos\theta$	$x = x(u,v,w)$
$y = y$	$y = r\sin\theta$	$y = \rho\sin\phi\sin\theta$	$y = y(u,v,w)$
$z = z$	$z = z$	$z = \rho\cos\phi$	$z = z(u,v,w)$

Area (2d) $\vec{\mathbf{r}} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$ $\qquad x = x(u,v), y = y(u,v)$

$$dA = J\,du\,dv \qquad J = |\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v| = \left\| \begin{matrix} x_u & y_u \\ x_v & y_v \end{matrix} \right\|$$

Rect Area $\qquad \boxed{dA = dy\,dx = dx\,dy}$

Polar Area $\qquad \boxed{dA = r\,dr\,d\theta}$

Volume (3d) $\vec{\mathbf{r}} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ $\qquad x = x(u,v,w), y = y(u,v,w), z = z(u,v,w)$

$$dV = J\,du\,dv\,dw \qquad J = |\vec{\mathbf{r}}_w \cdot (\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v)| = \left\| \begin{matrix} x_u & y_u & z_u \\ x_v & y_v & z_v \\ x_w & y_w & z_w \end{matrix} \right\|$$

Rect Volume $\qquad \boxed{dV = dz\,dy\,dx}$

Cyl Volume $\qquad \boxed{dV = r\,dz\,dr\,d\theta}$

Sph Volume $\qquad \boxed{dV = \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta}$

Arclength (1d) $y = f(x)$, $ds^2 = dx^2 + dy^2$, $dy = f'(x)\,dx$

$$y = f(x) \qquad ds = \sqrt{1 + f'(x)^2}\,dx$$

Surface Area (2d) $\vec{\mathbf{r}} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ $\qquad x = x(u,v), y = y(u,v), z = z(u,v)$ [1]

$$dS = |\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v|dA = \left\| \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix} \right\| du\,dv$$

Graph of surface $x = u, y = v, z = f(x,y)$, $\vec{\mathbf{r}} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + f(x,y)\hat{\mathbf{k}}$

$$z = f(x,y) \qquad \boxed{dS = \sqrt{1 + f_x^2 + f_y^2}\,dy\,dx}$$

Cylinder, radius $r = a$

$$\boxed{dS = a\,dz\,d\theta}$$

Sphere, radius $\rho = a$

$$\boxed{dS = a^2 \sin\phi\,d\phi\,d\theta}$$

Cone, angle $\phi = \alpha$

$$\boxed{dS = \rho\sin\alpha\,d\rho\,d\theta}$$

xy plane area, $x = x(u,v), y = y(u,v), z = 0$

$$dS = dA$$

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[1]Note: The "||stuff||" in the general formula for dS means "the length of the cross product vector given by the determinant."