# Math 103-03, Spring 2006, Exam 3

Name:

I have adhered to the Duke Community Standard in completing this examination.

Signature:

Instructions: You have 50 minutes. Calculators are not allowed. Always show all of your work. Partial credit is often awarded. Pictures are often helpful. Give simplified, exact answers, and make sure that they are clearly marked. The point value of each problem is roughly proportional to its page space.

Ask questions if any problem is unclear. In multipart problems, some parts depend on earlier parts and some do not. If you cannot solve a crucial part, on which later parts depend, then you may ask me for the answer to that part.

**1**. Consider the vector field $\vec{F} = \langle 2y, 2x + 2y \rangle$ on $\mathbb{R}^2$.
   A. Draw a sketch of $\vec{F}$.
   Solution: [Pick a point with integer coordinates, plug it into $\vec{F}$, and draw the resulting vector with its tail at the point you started with. Repeat for a bunch of points.]
   B. Find a potential function for $\vec{F}$, or show that none can exist.
   Solution: First, $\int 2y \, dx = 2yx + g(y)$; then $\frac{\partial}{\partial y}(2yx + g(y)) = 2x + g'(y) = 2x + 2y$, which implies that $g(y) = y^2 + C$. So the potential function is $f(x, y) = 2xy + y^2 + C$.

**2**. Let $\vec{F} = xy^6 \hat{\imath} + e^y \hat{\jmath} + \cos z \hat{k}$, a vector field on $\mathbb{R}^3$.
   A. Compute curl $\vec{F}$.
   Solution: curl $\vec{F} = \langle 0 - 0, 0 - 0, 0 - 6xy^5 \rangle = \langle 0, 0, -6xy^5 \rangle$.
   B. Let $\vec{G} = $ curl $\vec{F}$ be the answer to Part A. Compute $\iint_S \vec{G} \cdot \vec{n} \, dS$, where $S$ is the hemisphere $z = \sqrt{4 - x^2 - y^2}$ oriented with the upward-pointing choice of unit normal vector $\vec{n}$.
   Solution: The boundary curve $C$ is the circle $\vec{r}(t) = \langle 2\cos t, 2\sin t, 0 \rangle$ for $t$ running from 0 to $2\pi$; this is oriented compatibly with $\vec{n}$. Stokes' theorem says that

$$
\begin{aligned}
\iint_S (\text{curl } \vec{F}) \cdot \vec{n} \, dS &= \oint_C \vec{F} \cdot \vec{T} \, ds \\
&= \int_0^{2\pi} xy^6 \, dx/dt + e^y \, dy/dt + \cos z \, dz/dt \, dt \\
&= \int_0^{2\pi} (2\cos t)(2\sin t)^6(-2\sin t) + e^{2\sin t}(2\cos t) + 0 \, dt \\
&= \int_0^{2\pi} -128 \sin^7 t \cos t + e^{2\sin t}(2\cos t) \, dt
\end{aligned}
$$

$$= \left[-16\sin^8 t + e^{2\sin t}\right]_0^{2\pi}$$
$$= 0.$$

**3**. Let $T$ be the region enclosed by the cylinder $x^2 + y^2 = 4$, the plane $z = 0$, and the plane $z = 3$. Let $S$ be its surface (oriented with the outward-pointing normal $\vec{n}$). Let $\vec{F} = \langle y, yx^2, y^2 z + z^3/3 \rangle$. Compute the flux of $\vec{F}$ across $S$.

Solution: By the divergence theorem, the flux $\iint_S \vec{F} \cdot \vec{n}\, dS$ equals $\iiint_T \operatorname{div} \vec{F}\, dV = \iiint_T x^2 + y^2 + z^2\, dV$. Switching to cylindrical coordinates, the integral becomes

$$\int_0^3 \int_0^{2\pi} \int_0^2 (r^2 + z^2) r\, dr\, d\theta\, dz = \cdots = 60\pi.$$

**4**. Assume that the surface of the Earth is a perfect sphere of radius 6370 km. The tropical region is the area near the equator; technically, it extends from the Tropic of Cancer (at about $23.5°$ N latitude) to the Tropic of Capricorn (at about $23.5°$ S latitude). In order to save writing, let $\alpha = (90 - 23.5)(\pi/180)$ and $\beta = (90 + 23.5)(\pi/180)$; then, in spherical coordinates, the equator is at $\phi = \pi/2$, the Tropic of Cancer is at $\phi = \alpha$, and the Tropic of Capricorn is at $\phi = \beta$.

A. What fraction of the Earth's surface lies in the tropics? (Since you don't have a calculator, you will have to leave your answer in terms of $\alpha$ and/or $\beta$.)

Solution: On a sphere of radius 6370, the surface area element is $dS = 6370^2 \sin\phi\, d\phi\, d\theta$. So the area of the tropics is

$$\int_0^{2\pi} \int_\alpha^\beta 6370^2 \sin\phi\, d\phi\, d\theta = \cdots = 2\pi 6370^2 (\cos\alpha - \cos\beta) = 4\pi 6370^2 \cos\alpha$$

(since $\cos\phi$ is symmetric about $\phi = \pi/2$). The total area of the sphere is $4\pi 6370^2$. Therefore the fraction lying in the tropics is $\cos\alpha \approx 0.3987 = 39.87\%$.

B. In reality, the Earth is not a perfect sphere. Rather, it is slightly flatter than a sphere at its poles, and it bulges out slightly near the equator. Does the answer to Part A underestimate or overestimate the fraction of the Earth's surface lying in the tropics?

Solution: Part A underestimates the size of the tropics. Letting the tropics bulge out increases their distance from the rotational axis of the Earth, and therefore increases their surface area. To see this, imagine that the flattening/bulging is so extreme that the Earth looks like a pancake; then almost all of its surface lies in the tropics.