Math 103-03, Spring 2006, Exam 2 Answers

Instructions: DO NOT DO PAGE 6 (PROBLEM 5). The other problems earn points roughly proportional to their page space.

You have 50 minutes. Calculators are not allowed. Always show all of your work. Partial credit is often awarded. Pictures are often helpful. Give simplified, exact answers. Where space is provided for answers, write them there; otherwise, make sure they are clearly marked.

Ask questions if any problem is unclear. In multipart problems, some parts depend on earlier parts and some do not. If you cannot solve a crucial part, on which later parts depend, then you may ask me for the answer to that part.

1. Let S be the unit cube $[0,1] \times [0,1] \times [0,1]$ in the first octant of $\mathbb{R}^3$. Compute the following integral, which represents the average distance-squared to the origin over all points in S:

$$\iiint_S x^2 + y^2 + z^2 \, dV.$$

Solution: [This is an easier rewording of 14.6 #48.] The answer is 1. [I'll omit the work here.]

2. Heron's theorem says that the area a of a triangle with sides of length x, y, and z is

$$a = \sqrt{s(s-x)(s-y)(s-z)},$$

where $s = (x+y+z)/2$ is half the perimeter. Use this formula for the area to prove that, for any given nonzero perimeter $2s$, the triangle with the largest area is the equilateral one. (Hint: Maximize a^2 rather than a. Also, I recommend Lagrange multipliers, although other methods should work too.)

Solution: [This is 13.9 #36.] Let $f(x,y,z) = a^2 = s(s-x)(s-y)(s-z)$, where s is an unknown constant (half of the given perimeter). Let $g(x,y,z) = x+y+z$. We want to maximize $f(x,y,z)$ subject to the constraint that $g(x,y,z) = 2s$. Proceeding by Lagrange multipliers, we have

$$\nabla f = \langle -s(s-y)(s-z), -s(s-z)(s-x), -s(s-x)(s-y) \rangle$$

and $\nabla g = \langle 1,1,1 \rangle$. So we have equations

$$\begin{aligned} -s(s-y)(s-z) &= \lambda \\ -s(s-z)(s-x) &= \lambda \\ -s(s-x)(s-y) &= \lambda \\ x+y+z &= 2s. \end{aligned}$$

Multiplying the first three equations by $s-x$, $s-y$, and $s-z$, respectively, yields

$$\lambda(s-x) = \lambda(s-y) = \lambda(s-z).$$

If $\lambda = 0$, then two of x, y and z must equal s, which means that the other is 0, producing an area of 0 — clearly not the maximum. Thus $\lambda \neq 0$ and we conclude that $x = y = z$. This is a maximum, since the area decreases to 0 as x, y, or z goes to 0.

3. Let R be the region in the x-y-plane bounded by the lines $y = 0$ and $y = 1$ and the hyperbola $x^2 - y^2 = 1$. Let S be the solid region in $\mathbb{R}^3$ obtained by revolving R about the x-axis. In this two-page problem, you will compute the volume of S in two completely different ways.

Visualization notes: The region R is symmetric about the y-axis. The boundary of S consists of a cylindrical part and two hyperboloid caps (a hyperboloid of two sheets). If S were made of glass, it would be a concave lens, somewhat like the lenses that nearsighted people wear.

A1. Express the hyperboloid as the graph of x in terms of y and z.

Solution: $x = \pm\sqrt{1 + y^2 + z^2}$.

A2. Compute the volume of S using part A and polar coordinates $y = r\cos\theta$ and $z = r\sin\theta$ in the y-z-plane.

Solution: Let $y = r\cos\theta$ and $z = r\sin\theta$, so that the hyperboloid is $x = \pm\sqrt{1 + r^2}$. We wish to integrate over the unit disk R in the y-z-plane. The integral is

$$\iint_R 2\sqrt{1 + y^2 + z^2}\; dx\; dy = \int_0^{2\pi} \int_0^1 2\sqrt{1 + r^2}\; r\; dr\; d\theta.$$

Substituting $u = 1 + r^2$ leads to an answer of $4\pi(2^{3/2} - 1)/3$.

B1. Compute the centroid (center of mass) of R. Assume that R has constant areal density $\delta \equiv 1$. The area a of R is not easy to compute, so DO NOT COMPUTE THE AREA; give your answer in terms of a, if you must.

Solution: The x-coordinate of the centroid is $\bar{x} = 0$, by symmetry. The y-coordinate $\bar{y}$ is

$$\bar{y} = \frac{1}{a}\iint_R y\; dA = \int_0^1 \int_{-\sqrt{1+y^2}}^{\sqrt{1+y^2}} y\; dx\; dy.$$

The integration dx is trivial, and the integration dy is easily done by substituting $u = 1 + y^2$. The answer is $2(2^{3/2} - 1)/(3a)$.

B2. Compute the volume of S using part C and a theorem of Pappus (in terms of a, if you must).

Solution: The centroid is $2(2^{3/2} - 1)/(3a)$ units from the origin, so it travels a distance of $4\pi(2^{3/2} - 1)/(3a)$ during the revolution. So Pappus says that the volume of the solid is $a4\pi(2^{3/2} - 1)/(3a) = 4\pi(2^{3/2} - 1)/3$. This matches our answer from part B.

4. A snake is slowly slithering through a flat jungle (the x-y-plane). Her path is described by the parametrized plane curve $\vec{r}(t) = \langle x(t), y(t)\rangle$. Parts of the jungle are warm, and other parts are cool; the temperature z at any point (x, y) is given by the function $z = f(x, y)$. Since the snake is cold-blooded, her temperature always equals the ambient temperature around her. The quantity dz/dt then describes the rate of change of the snake's temperature with respect to time.

A. Using the chain rule, derive a formula for dz/dt in terms of the gradient $\nabla f(x, y)$ and the derivative $\vec{r}\,'(t)$. (This part is not needed for parts B and C.)

Solution: By the chain rule,

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = \left\langle \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y} \right\rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle = \nabla f(x, y) \cdot \vec{r}\,'(t).$$

B. Suppose that $f(x, y) = x^2/2 + y^2$ and that the snake is at $(4, 0)$. Describe the path that she should take, to keep her temperature constant.

Solution: The level sets of f are ellipses. The snake's level set is the ellipse $x^2/2+y^2 = 8$, so she should move along this ellipse. You are not required to give $\vec{r}(t)$ explicitly, but here is one answer: $\vec{r}(t) = \langle 4\cos t, 2\sqrt{2}\sin t\rangle$.

C. Suppose that $f(x, y) = xy$ and that the snake is at the origin. Describe the path that she should take, to keep her temperature constant.

Solution: The level set through the origin is the union of the x- and y-axes. She should stay on the axes.

5. DO NOT DO THIS PAGE. Consider the integral

$$\int_{-2}^{2}\int_{0}^{\sqrt{1-x^2/4}} x^2 \, dy \, dx.$$

A. Sketch the region of integration.

Solution: [It's the top half of the region bounded by the ellipse $x^2/4 + y^2 = 1$.]

B. Reexpress the integral as an integral $dx\, dy$. Do not evaluate the integral.

Solution: The integral is

$$\int_{0}^{1}\int_{-2\sqrt{1-y^2}}^{2\sqrt{1-y^2}} x^2 \, dx \, dy.$$

C. Reexpress the integral in polar coordinates. Do not evaluate the integral.

Solution: In polar coordinates, the ellipse in question is $r^2 \cos^2\theta + 4r^2 \sin^2\theta = 4$. Solving for r, we obtain $r = 2/\sqrt{1 + 3\sin^2\theta}$. Also, the integrand x^2 becomes $r^2 \cos^2\theta$. Therefore the integral is

$$\int_0^\pi \int_0^{2/\sqrt{1+3\sin^2\theta}} r^3 \cos^2\theta \, dr \, d\theta.$$