Math 103-03, Spring 2006, Exam 1

Name:

I have adhered to the Duke Community Standard in completing this examination.

Signature:

Instructions: You have 50 minutes. Calculators are not allowed. Always **show all of your work**. Pictures are often helpful. Partial credit may be awarded. Give **simplified**, **exact** answers, and make sure they are clearly marked.

1 (12 pts). Let $\vec{u} = \langle 3, \pi, -1\rangle, \vec{v} = \langle 2, 1, -3\rangle$ be vectors in $\mathbf{R}^3$. Compute the following quantities. Write your final answers in the spaces provided. (Here $\mathrm{comp}_{\vec{v}}\vec{u}$ is as defined in the book; it is the length of the vector $\vec{u}^{||} = \mathrm{proj}_{\vec{v}}\vec{u}$ discussed in class.)

$\vec{u} + \vec{v} =$ ____________

$2\vec{u} =$ ____________

$6\vec{u} + -4\vec{v} =$ ____________

$\vec{u} \cdot 2\vec{v} =$ ____________

$\mathrm{comp}_{\vec{v}}\vec{u} =$ ____________

$\vec{u} \times \vec{v} =$ ____________

2 (8 pts). Let $\vec{u}$, $\vec{v}$, $\vec{w}$ be any three nonzero vectors in $\mathbf{R}^3$. Under what conditions will it be true that $|\vec{u} \times (\vec{v} \times \vec{w})| = |\vec{u}||\vec{v}||\vec{w}|$? Explain in detail, including the geometric meaning of your answer.

3 (8 pts). Let $\vec{a}$ and $\vec{b}$ be two distinct points in $\mathbf{R}^3$. (Here, as usual, we are identifying a vector with the point it points to when its tail is at the origin.) The *perpendicular bisector* of the line segment from $\vec{a}$ to $\vec{b}$ is the plane through the midpoint of the line segment that is perpendicular to the line segment. Find an equation for the perpendicular bisector.

4 (12 pts). An airplane on a mission to drop humanitarian aid packages over a war-torn region is flying directly east at a speed of 40 m/s and at an altitude of 1000 meters. Suddenly the copilot spots a needy person named Trevor directly below the plane and releases a package for him. The package accelerates down at 10 m/s^2 due to gravity. A wind blowing southeast also accelerates the package at 2 m/s^2. How far from Trevor does the package land? Write your answer in the space provided.

distance = ____________

5 (12 pts). Let $f(x, y) = \frac{y^2x^3}{y^4+x^8}$.

A. What does the limit $\lim_{(x,y)\to(0,0)} f(x, y)$ appear to be if we approach the origin along the x-axis? Along the y-axis? Along any other line $y = mx$? Write your answers in the spaces provided.

x-axis: ____________

y-axis: ____________

$y = mx$: ____________

B. Can you conclude that the limit exists, or that it does not exist, or neither? Why?

6 (20 pts). The figure below shows the graph of the parametric plane curve

$$\vec{r}(t) = \langle x(t), y(t)\rangle = \langle t(t-1)^2, t^2(t-1)\rangle = \langle t^3 - 2t^2 + t, t^3 - t\rangle).$$

A. On the graph, clearly mark all points where $x'(t) = 0$ and all points where $y'(t) = 0$. (You do not need to give their coordinates.)

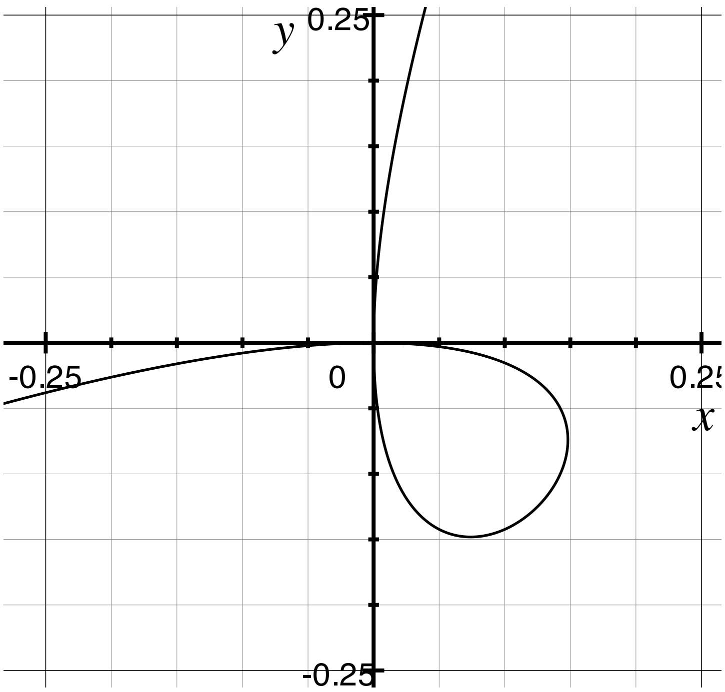
B. Compute the speed v, unit tangent vector $\vec{T}$, curvature κ, and unit normal vector $\vec{N}$ at time $t = 1/2$. Write your final answers in the spaces provided, and draw $\vec{T}$ and $\vec{N}$ on the graph.

$v =$ ____________

$\vec{T} =$ ____________

$\kappa =$ ____________

$\vec{N} =$ ____________



7 (12 pts). In this problem you will find the point on the surface $x^2y^2z = 4$ (with $x, y, z > 0$) that is closest to the origin.

A. What function are you going to minimize? (If you do not know then you may ask me and I will tell you a function so that you can attempt parts B and C.)

B. On what region R are you minimizing this function? Is R a closed, bounded region (the sort that we like to minimize on)? How do you know that you will find a minimum?

C. Find the minimum. That is, what is the point on the surface in the first octant that is closest to the origin?

8 (8 pts). Mathematician Emmy Noether (1882 - 1935) is driving her motorcycle due south through scenic Germany. The surface of the land around her is described by the graph of

$$z = e^{x^2 - y^2},$$

with the positive x- and y-axes pointing east and north, as usual. When she reaches the point $(0, 1, 1/e)$, she suddenly turns off gravity (she's that good) and goes flying off tangentially to the surface. Describe her flight trajectory as a parametrized line.

9 (8 pts). On a recent trip to suburban Madagascar I met a UNC professor who claimed that her favorite function $f(x, y)$ had partial derivatives

$$f_x = \frac{-y}{\sqrt{x^2 + y^2}}, \quad f_y = \frac{x}{\sqrt{x^2 + y^2}}.$$

I was skeptical that such a function $f(x, y)$ could even exist. She conceded that $f(x, y)$ was not defined at the origin, but asserted that it was defined everywhere else.

What do you think: Does such a function exist? Explain your evidence for or against it. (You are not required to find the function.)