Geology 376-2, Spring 2004, Assignment 2

You are encouraged to work with others (and to ask me questions), but you should compose and submit your solutions independently. Give exact answers, and show your work. Let me know if you find any errors.

1. Let $v = \langle v_1, v_2, v_3 \rangle$ and $w = \langle w_1, w_2, w_3 \rangle$ be two vectors in in $\mathbb{R}^3$. Recall that their dot and cross products are defined as

$$\begin{aligned} v \cdot w &= v_1 w_1 + v_2 w_2 + v_3 w_3, \\ v \times w &= \langle v_2 w_3 - v_3 w_2, v_3 w_1 - v_1 w_3, v_1 w_2 - v_2 w_1 \rangle. \end{aligned}$$

A. Show that $v \times w = -(w \times v)$. (This is called *anti-* or *skew-symmetry.*)

B. Which one makes sense, $w \cdot (v \times w)$ or $(w \cdot v) \times w$? Why?

C. Show that the one in part B that exists is equal to 0. What is the geometric significance of its equalling 0?

2. Consider, ladies and gentlemen, the linear system

$$\begin{aligned} -2x + 4y &= 8 \\ x + 6y + 12z &= 12 \\ 2x - 3y + 3z &= -9 \end{aligned}$$

A. Express this linear system as a matrix equation of the form $AX = B$. (That is, tell me what A, X, and B should be.)

B. Solve the equation by inverting A using the adjoint.

C. Solve the equation by inverting A using Gaussian elimination. (That's the method in which you juxtapose the matrix with the identity and perform row operations until you get the identity on the left and the inverse on the right).

D. Solve the equation by another kind of Gaussian elimination: Juxtapose A with B, forming a 3×4 matrix, and then perform row operations until you obtain the identity on the left; then the answer X will appear on the right.

3. Suppose we want to solve two linear systems, $AX = B$ and $AY = C$, involving the same invertible matrix A. Then the methods of parts 2B and 2C seem more efficient than that of part 2D, because the method of 2D requires us to do the work of inverting A twice, once for each equation, even though the inverse of A has nothing to do with B and C.

However, when A is not invertible, the methods of parts 2B and 2C don't work, while the method in part 2D still gives us useful information — information that depends dramatically on B and C. For this problem, we will work with

$$A = \begin{bmatrix} 3 & 6 & -9 \\ -1 & -2 & 4 \\ 7 & 14 & -22 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

A. Compute the determinant of A.

B. Attempt to solve $AX = B$ using the method of part 2D. You will not succeed in obtaining the identity matrix on the left; the closest you can get is

$$\begin{bmatrix} 1 & 2 & 0 & ? \\ 0 & 0 & 1 & ? \\ 0 & 0 & 0 & ? \end{bmatrix},$$

where the ? are the components of some vector (not necessarily equal). Now reinterpret this 3×4 matrix as a linear system of equations. This system has exactly the same solutions as the original $AX = B$ in this problem, but it is simpler, so we have made progress, right? Briefly describe the solution set, geometrically, and give me at least two solutions.

C. Following the same procedure, attempt to solve $AX = C$. What happens? Why?

D. For which vectors D does $AZ = D$ have a solution Z? What does this have to do with the range of A? What is the rank of A?

E. What is the kernel of A?